John von Neumann (1903-1957)

- Child prodigy (by 12, finished Borel's function theory)
- Most pure & applied contributions in math than any other mathematician
  - Math: functional analysis\operator algebras, ergodic group theory, game theory...
  - Economics
  - Computer science
  - Physics (quantum mechanics)

History of vN's work in functional analysis

- In 1920's, Heisenburg, Jordan were working on a theory of infinite matrices ("matrix mechanics")
- In 1930's, v.N. formalized and greatly expanded this theory by introducing Hilbert spaces, operators on them, spectral theory, and studying algebras of these operators (with Murray: On Rings of Operators I-IV).

He introduced "Dirac-von Neumann axioms of Quantum Mechanics."

In our class, we work over $\mathbb{C}$. So, Hilbert spaces have the inner-product $\mathbb{C}$-valued (so it satisfies $\langle x,y\rangle = \overline{\langle y,x\rangle}$ and $\langle c_1x, c_2y\rangle = c_1\bar{c_2}\langle x,y\rangle$).

Example: • On $\mathbb{C}^n$, $\langle (x_1,\dots,x_n),(y_1,\dots,y_n)\rangle = \sum x_i\overline{y_i}$.

$B^* = \overline{B}^T$

- On $M_n(\mathbb{C})$, $\langle A,B\rangle = \mathrm{Tr}(AB^*) = \sum_{k=1}^n (AB^*)_{kk} = \sum_{k=1}^n \sum_{i=1}^n (A)_{ki}(B^*)_{ik} = \sum_{i,k=1}^n a_{ki}\overline{b_{ki}}$.
- $\ell^2(\mathbb{N}) = \{(x_n)_{n\in\mathbb{N}} : x_n\in\mathbb{C},\ \sum|x_n|^2<\infty\}$, $\langle (x_n),(y_n)\rangle = \sum x_n\overline{y_n}$.
- $L^2(X,\mu) = \{f:X\to\mathbb{C} : \int|f|^2d\mu<\infty\}$ w/ (X,μ) measure space, $\langle f,g\rangle = \int f\bar{g}\,d\mu$.

Remark: $\overset{\text{Lebesgue}}{L^2([0,1],m)} \cong \ell^2(\mathbb{N})$.

Recall: A Hilbert space H admits ONB's: an ONB is $(e_i)_{i\in I} \in H$ ∋ $\begin{cases} e_i \text{ are orthonormal: } \|e_i\|=1,\ \langle e_i, e_j\rangle = 0 \text{ if } i\neq j. \\ \overline{\operatorname{span}\{e_i : i\in I\}} = H \end{cases}$

In this case, if $x\in H$, $x=\sum_{i\in I} c_i e_i$ w/ $c_i = \langle x, e_i\rangle$ (the Fourier coeff. of x).

We will work mainly with <u>separable</u> Hilbert spaces ($|I|$ at most countable).

Example: $H = L^2([0,1],m)$. Let $e_n(x) = e^{2\pi i n x}$ $\forall n\in\mathbb{Z}$. **Claim:** $\{e_n\}_{n\in\mathbb{Z}}$ form an ONB for H.

Proof: It is easy to check that $\|e_n\|=1$ and $\langle e_k, e_\ell\rangle = 0$ $\forall k\neq \ell$ (i.e. $\int_0^1 e^{2\pi i k x} e^{-2\pi i \ell x}\,dx = 0$ $\forall k\neq\ell$).

Why is $\overset{=A}{\operatorname{span}\{e_k\}}$ dense in L^2?

$A\subset C([0,1]) \subset L^2([0,1])$ (dense in $C([0,1]) \Rightarrow$ dense in $L^2([0,1])$, since $C([0,1])$ is dense in $L^2([0,1])$).

Stone-Weierstrass Thm:

If $A\subset C([0,1])$ satisfies:

(1) A is a unital $\mathbb{C}$-algebra

(2) A is closed to conjugates: if $f\in A$, then $\bar{f}\in A$

(3) A separates the points: $\forall x,y\in[0,1]$, $\exists f\in A$ ∋ $f(x)\neq f(y)$.

Then $\overline{A}^{\|\cdot\|_\infty} = C([0,1])$.

So, for our A it follows $\overline{A}^{\|\cdot\|_\infty} = C([0,1])$. Also, $\overline{C([0,1])}^{\|\cdot\|_2} = L^2([0,1])$. Thus, $\overline{A}^{\|\cdot\|_2} = \overline{\overline{A}^{\|\cdot\|_\infty}}^{\|\cdot\|_2} = \overline{C([0,1])}^{\|\cdot\|_2} = L^2([0,1])$.

Remark: There is only one (up to iso.) Hilbert space in each dimension: $\dim H = |I|$, if $(e_i)_{i\in I}$ is an ONB for H.

So, $L^2([0,1],m) \cong \ell^2(\mathbb{N})$.

ONB's: $(e^{2\pi i k})_{k\in\mathbb{Z}}$, $(\delta_n)_{n\in\mathbb{N}}$ w/ $\delta_n = (0,0,\dots,0,\overset{n}{1},0,\dots)$.