Useful Formulas in Hilbert spaces

- Parallelogram rule: $\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2$

- Polarization: $\langle x,y\rangle = 1/4\sum_{k=0}^{3} i^k\|x+i^k y\|^2$

If T is a bounded operator, $\langle Tx,y\rangle = 1/4\sum_{k=0}^{3} i^k\langle T(x+i^k y), x+i^k y\rangle$. So, if we know $\langle Tv,v\rangle$ $\forall v\in H$, we know T.

So, if $\langle Tv,v\rangle=0$ $\forall v\in H$, then $T=0$.

$B(H)=\{T:H\to H$ bounded operator$\}$. $(B(H),\|\ \|)$ is a Banach $*$-algebra, T^* = adjoint of T.

Properties of operator norm

- $\|T\|=0 \Leftrightarrow T=0$, T subadditive $\quad \forall T,S\in B(H)$

- $\|TS\|\le\|T\|\|S\|$

- $\|T^*\|=\|T\|$

self-adjoint

- $\|T^*T\|=\|T\|^2$ ← C^* property

Let's prove last two: Review $\|T\|=\sup_{\|x\|,\|y\|\le1}|\langle Tx,y\rangle|$

$\|T^*T\|\le\|T^*\|\|T\|$. Also, $\|T^*T\|=\sup_{\|x\|,\|y\|\le1}|\langle T^*Tx,y\rangle|=\sup_{\|x\|,\|y\|\le1}|\langle Tx,Ty\rangle|\le\|Tx\|\|Ty\|\le\|T\|\|T\|$, hence $\|T^*T\|\le\|T\|^2$.

$\|T\|^2=\sup_{\|x\|\le1}\|Tx\|^2=\sup_{\|x\|\le1}\langle Tx,Tx\rangle=\sup_{\|x\|\le1}\langle T^*Tx,x\rangle\le\|T^*Tx\|\|x\|\le\|T^*T\|$. $\Rightarrow \|T^*T\|=\|T\|^2$.

$\|T\|^2=\|T^*T\|\le\|T^*\|\|T\| \Rightarrow \|T\|\le\|T^*\|$. Using T^* gives $\|T^*\|\le\|T\|$.

Examples of operators

- $H=\mathbb{C}^n$. Any matrix $A\in M_n(\mathbb{C})$ gives $T=L_A$ (left mult. by A), $T(x)=Ax$. Exercise

right shift

- $H=\ell^2(\mathbb{N})$. $S:H\to H$ left shift, $S(x_1,x_2,x_3,\dots)=(x_2,x_3,x_4,\dots)$. $S^*(x_1,x_2,x_3,\dots)=(0,x_1,x_2,\dots)$. $\quad \varphi\leftrightarrow(c_1,\dots,c_n)$

Note $SS^*\neq S^*S$, as $SS^*=I$ and $S^*S\neq I$. $\quad f\leftrightarrow(x_1,\dots,x_n)$, $M_\varphi(f)=(c_1x_1,\dots,c_nx_n)$.

- Let (X,μ) be a measure space, $H=L^2(X,\mu)$, and $\varphi\in L^\infty(X,\mu)$. Let $T=M_\varphi$ be given by $T(f)=\varphi f$ $\forall f\in H$.

$|\varphi| \le \|\varphi\|_\infty$

Why is T bounded? $\|Tf\|^2 = \|M_\varphi f\|^2 = \int |\varphi f|^2 \le \|\varphi\|_\infty^2 \int |f|^2 = \|\varphi\|_\infty^2 \|f\|^2$. So, $\|T\| \le \|\varphi\|_\infty$. In fact:

<u>Thm:</u> If (X,μ) is σ-finite, then $\|M_\varphi\| = \|\varphi\|_\infty$. $\varphi \to M_\varphi$, $L^\infty(X,\mu) \hookrightarrow \mathcal{B}(L^2(X,\mu))$

$\Leftrightarrow \|M_\varphi f\|_2 \ge c\|f\|_2$

<u>Proof:</u> $\|M_\varphi\| \le \|\varphi\|_\infty$ done. Let $0 < c < \|\varphi\|_\infty$. We create a vector $f \in L^2(X,\mu)$ w/ $\|M_\varphi f\|_2 \ge c$, $\|f\|_2 = 1$.

$\int |f|^2 = \mu(E)$

Try $f = \chi_E$ (E meas. in X, $\mu(E) \ne 0$). $\|M_\varphi f\|_2^2 \ge c^2 \|f\|_2^2 \Leftrightarrow \int_X |\varphi f|^2 \ge c^2 \mu(E) \Leftrightarrow \int_E |\varphi|^2 \ge c^2 \mu(E)$. Let $E \subseteq \{x \in X : |\varphi(x)| \ge c\}$ s.t. $0 < \mu(E) < \infty$

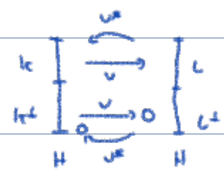
(exists by σ-finiteness). So, $\int_E |\varphi|^2 \ge \int_E c^2 = c^2 \mu(E)$.

<u>Important classes of operators</u>

(1) $T \in \mathcal{B}(H)$ is self-adjoint if $T = T^*$

(2) $N \in \mathcal{B}(H)$ is normal if $N^*N = NN^*$

(3) $U \in \mathcal{B}(H)$ is unitary if $U^*U = UU^* = I$. Equivalently, $\langle Ux, Uy \rangle = \langle x, y \rangle$ $\forall x, y \in H$.



(4) $V \in \mathcal{B}(H)$ is an isometry if $V^*V = I$. Equivalently, $\|Vx\| = \|x\|$ $\forall x \in H$.

(5) $V \in \mathcal{B}(H)$ is a partial isometry (p.i.) if $VV^*V = V$. Equivalently, $\exists K \subseteq H$ closed subspace w/ $V|_K$ isometric and $V|_{K^\perp} = 0$.

Equivalently, V^*V, VV^* are projections (and $V^*V = \mathrm{proj}_K$).

(6) $P \in \mathcal{B}(H)$ is a projection if $P = P^2 = P^*$. Equivalently, there is $K \subseteq H$ closed subspace w/ $P = \mathrm{proj}_K$, meaning

$\forall x \in H$, $y = P(x)$ attains the $\inf_{y \in K} \|y - x\|$.