Important Classes of Operators in $B(H)$

- **Isometry:** $V \in B(H)$ w/ $\|V\xi\| = \|\xi\|$ $\forall \xi \in H$

$\iff \|V\xi\|^2 = \|\xi\|^2 \iff \langle V\xi, V\xi\rangle = \langle \xi, \xi\rangle$

$\iff \langle V^*V\xi, \xi\rangle = \langle \xi, \xi\rangle \quad \forall \xi$

$\iff \langle (V^*V - I)\xi, \xi\rangle = 0 \quad \forall \xi$

polarization

$\iff \langle (V^*V - I)\xi, \eta\rangle = 0 \quad \forall \xi, \eta \in H$

$\iff V^*V = I$

$\iff \langle V\xi, V\eta\rangle = \langle \xi, \eta\rangle.$

- **Unitary:** $U \in B(H)$ $\ni$ U is isometry and $\overset{\text{surjection}}{\downarrow}$ bijection.

so $U^*U = I = UU^*$.

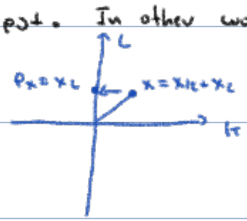
Remark: If V is an isometry, then the range of V (VH) is closed.

Indeed, if $\eta \in \overline{VH}$, $\eta = \lim_{n\to\infty} V\xi_n$, $\xi_n \in H$. $\Rightarrow V\xi_n$ is Cauchy $\Rightarrow \|V\xi_n - V\xi_m\|$ "gets small" $\Rightarrow \|\xi_n - \xi_m\|$ "gets small"

$\Rightarrow \xi_n$ is Cauchy $\Rightarrow \xi_n \to \xi \in H \Rightarrow V\xi_n \to V\xi = \eta \in VH$.

- **Projections:** $P \in B(H)$ is a projection if $P|_{(\ker P)^\perp} = id|_{(\ker P)^\perp}$. In other words, if $K = \ker P$ and $L = (\ker P)^\perp$,

then $H = K \oplus L$ and $P|_K = 0$ and $P|_L = id_L$.



Thm: P proj. $\iff P = P^2 = P^*$.

Exercise: Say $P = P_K$, $Q = P_L$ w/ K, L closed subspaces of H. In terms of K and L, characterize each property:

(i) $PQ = 0$ (iii) $PQ = P$

(ii) $P+Q$ is a projection (iv) PQ is a projection

(i) $pq=0 \Rightarrow P_kP_l=0 \Rightarrow P_kP_l(L)=0 \Rightarrow P_k(L)=0 \Rightarrow L\subset \ker P_k = k^\perp \Rightarrow L\perp k$. If $k\perp L$, then $P_kP_l\xi \in P_k(L)=0 \Rightarrow P_kP_l=0$.

$\quad$ ← already self-adjoint $\qquad\qquad p\cdot \qquad \cdot p \qquad k\perp L$

(ii) $p+q$ proj. $\Leftrightarrow (p+q)^2=p+q \Leftrightarrow p+q+pq+qp=p+q \Leftrightarrow pq+qp=0 \Rightarrow pq+pqp=0,\ pqp+qp=0 \Rightarrow pq=qp=0 \Rightarrow p\perp q$.

If $p\perp q$, then $p+q=P_{k+L}$.

take *

(iii) $P_kP_l=P_k \Leftrightarrow P_lP_k=P_k \Leftrightarrow P_l(\xi)=\xi\ \ \forall \xi\in k \Leftrightarrow k\subset L$ (b.c. P_l is id on L only).

We write $p\leq q$ if $pq=p$.

(iv) pq projection $\Rightarrow pq=(pq)^*=qp$ and $(pq)^2=pq$ (which follows from $pq=qp$). So, pq proj. $\Leftrightarrow pq=qp$. In this case,

$P_kP_l\xi = P_lP_k\xi \in k, L \Rightarrow P_kP_l L \in k\cap L$. In fact, P_kP_l has range $k\cap L$, as $\xi\in k\cap L \Rightarrow P_kP_l\xi=\xi$. Also, P_kP_l is a proj.,

so $P_kP_l = P_{k\cap L}$. $\Leftrightarrow P_kP_l = P_kP_{k\cap L} \Leftrightarrow P_k(P_l-P_{k\cap L})=0 \Leftrightarrow P_kP_{L\ominus k\cap L}=0 \Leftrightarrow k\perp L\ominus(k\cap L) \Leftrightarrow k\ominus(k\cap L)\perp L\ominus(k\cap L)$.

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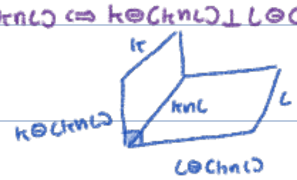
$L\ominus k\cap L = L\cap(k\cap L)^\perp$ (note $H=k\oplus k^\perp$, so $H\ominus k = k^\perp$)



• Partial isometry: $v\in B(H)$ is a p.i. (partial isometry) if $v|_{(\ker v)^\perp}$ is an isometry.

So, $H=k\oplus L$, $k=\ker v$, $L=k^\perp$ w/ $v|_k=0$, $v|_L$ isometry.

Example: $H=\mathbb{C}^2$, $v\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}0\\x\end{pmatrix}$. v is multiplication by matrix $\begin{pmatrix}0&1\\0&0\end{pmatrix}=e_{12}$.

$\ker v=\left\{\begin{pmatrix}0\\y\end{pmatrix}: y\in\mathbb{C}\right\}$, $\ker v^\perp=\left\{\begin{pmatrix}x\\0\end{pmatrix}: x\in\mathbb{C}\right\}$. v on $\ker v^\perp$ is isometric: $\begin{pmatrix}x\\0\end{pmatrix}\overset{\sim}{\mapsto}\begin{pmatrix}0\\x\end{pmatrix}$.

Let's discover more properties of a p.i. v.

$$\begin{array}{ccc} k & \xrightarrow{v} & \{0\}\in k'=(L')^\perp \\ \oplus & & \oplus \\ L & \xrightarrow{v} & L'=vL \text{ (closed)} \end{array} \qquad \text{Expect: } \begin{array}{ccc} \{0\} & \xleftarrow{v^*} & k' \\ & & \oplus \\ L & \xleftarrow{v^*} & L' \end{array}$$

Why is $v^*(k')=0$? Let $\eta\in k'^\perp$, so $\eta\perp v\xi\ \ \forall\xi\in H$. $\langle v^*\eta,\xi\rangle=\langle\eta,v\xi\rangle=0$, so $v^*\eta\perp\xi\ \ \forall\xi\in H \Rightarrow v^*\eta=0$.

Why is $v^*(L')\subset L$? Let $v\xi\in L'$ ($\xi\in L$). WTS: $v^*(v\xi)\in L$. We show $v^*(v\xi)\perp L^\perp=k$, i.e. $\langle v^*(v\xi),\eta\rangle=0\ \ \forall\eta\in k$.

$\Leftrightarrow \langle v\xi, v\eta\rangle=0$ (true b.c. $v\eta=0$).

In fact, $u^*u|_C = \mathrm{id}|_C$. ← Exercise

<u>Consequence:</u> $u^*uu^* = u^*$ ($\xi \in H \Rightarrow u^*\xi \in C$, so $u^*uu^*\xi = u^*\xi$).