646: vN alg. GNS construction, Spectral Thm ⊕ Review spectral thm

642: Banach, C^*-alg. Spectral Thm

Last time: $u: H \to H$ is a p.i. if $u|(\ker u)^\perp$ is an isometry.

We saw that u p.i. $\Rightarrow$ $u^*uu^* = u^*$.

Thm: TFAE for $u \in B(H)$: $\qquad \ker u = K \qquad K' = \ker u^*$

$$\overset{\oplus}{L} \underset{u^*}{\overset{u}{\rightleftarrows}} \overset{\oplus}{L'} = u(H)$$

(1) u is a p.i. $\qquad$ (4) $uu^*u = u$

(2) u^* is a p.i. $\qquad$ (5) u^*u projection $\qquad u^*u = P_L,\ uu^* = P_{L'}$

(3) $u^*uu^* = u^*$ $\qquad$ (6) uu^* projection

Proof: (1) $\Rightarrow$ (3) done. (3) $\Leftrightarrow$ (4) (taking adjoints).

(3) $\Rightarrow$ (5) u^*u is self-adjoint and $(u^*u)^2 = u^*uu^*u = u^*u$. (4) $\Rightarrow$ (6) similarly.

Why does (5) imply (3) or (4)? (5) tells us $(u^*u)^2 = u^*u$, or $u^*uu^*u = u^*u$. WTS: $u^*uu^* = u^*$ or $uu^*u = u$.

Let $s = uu^*u - u$. WTS: $s = 0$. $\Leftrightarrow$ $s^*s = 0$, as $\|s\|^2 = \|s^*s\|$. Indeed, $s^*s = (u^*uu^* - u^*)(uu^*u - u) = (u^*u)^3 + u^*u - (u^*u)^2 - (u^*u)^2 = 0$.

Know: (1) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). Why do (3)-(6) imply (1)?

$u^*u = P_L$ w/ $L^\perp = \ker P_L = \ker u^*u = \ker u$. So, u is an isometry on $(\ker u)^\perp = L$ (as $u^*u|_L = id|_L$).

$\ker T^*T = \ker T$, as $T^*T\xi = 0 \Rightarrow \langle T^*T\xi, \xi\rangle = 0 \Rightarrow \langle T\xi, T\xi\rangle = 0 \Rightarrow T\xi = 0$.

So, (1)-(6) are equivalent. u^* is a p.i. b.c. $(u^*)^*(u^*)(u^*)^* = (u^*)^*$ $\Leftrightarrow$ $uu^*u = u$.

Thm. (Polar Decomposition)

Let $T \in B(H)$. There exists p.i. u $\ni$ $T = u|T|$ and $\underset{\text{uniqueness}}{\ker u = \ker T}$ $(= \ker|T|)$. $\qquad$ Think $z = e^{i\theta}|z|$

Sketch of proof: Recall $|T| = (T^*T)^{1/2}$. Note $\ker|T| = \ker T = \ker T^*T$, as $|T|\xi = 0 \Leftrightarrow \langle |T|\xi, |T|\xi\rangle = 0 \Leftrightarrow \langle |T|^2\xi, \xi\rangle = 0 \Leftrightarrow \langle T^*T\xi, \xi\rangle = 0$

$\Leftrightarrow \langle T\xi, T\xi\rangle = 0 \Leftrightarrow T\xi = 0$.

same norm

$(\ker T^*)^\perp = \overline{\mathrm{range}(T)}$, $|T| = |T|^*$

<u>Want</u>: $T\xi = u|T|\xi$. So, define by $u(|T|\xi) = T\xi$. u is an iso. on $\overline{|T|(H)} = \ker|T|^\perp$. Define $u = 0$ on $\ker|T|$.

<u>Continuous Functional Calculus</u>

<u>Thm</u>: Let $T \in B(H)$ normal. There is a unique norm-preserving $*$-isomorphism between $\mathcal{A} = C^*(\{1, T\})$ (the smallest norm-closed unital $*$-alg of $B(H)$ containing T = $\overline{\{p(T, T^*) : p \in \mathbb{C}[x,y]\}}$ " ") and $C(\sigma(T))$ such that it takes T to $id_{\sigma(T)}$.

Let's untangle this isomorphism Φ. $C^*(\{1, T\}) \xrightarrow{\Phi} C(\sigma(T)) = \{f(z) \text{ cont.}, \mathbb{C}\text{-valued}\}$

$\lambda \in \mathbb{C} \longleftrightarrow$ constant function λ $\qquad e^T \longleftrightarrow e^z$

$T \longleftrightarrow id_{\sigma(T)}(z) = z$ $\qquad |T| \longleftrightarrow |z|$

$T^2 \longleftrightarrow z^2$ $\qquad \sqrt{T} \longleftrightarrow \sqrt{z}$ only makes sense if $\sigma(T) \subset [0, \infty)$

$T^3 \longleftrightarrow z^3$

$T^* \longleftrightarrow \bar{z}$

$T^2 T^* \longleftrightarrow z^2 \bar{z}$

<u>Remark</u>: For any $f \in C(\sigma(T))$, we can define $\Phi^{-1}(f) = f(T)$.

<u>Recall</u>: $\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ not invertible}\} \subset \mathbb{C}$ compact

Say $H = \mathbb{C}^n$. $T\xi = A\xi$ w/ $A \in M_n(\mathbb{C})$. $T^*\xi = A^*\xi$ w/ $A^* = \overline{A}^T$.

If T normal (so $A^*A = AA^*$), we know $A = UDU^*$ w/ U unitary, $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. $\sigma(T) = \sigma(A) = \sigma(D) = \{\lambda_1, \dots, \lambda_n\}$.

$C^*(\{1, T\}) \cong \{p(A, A^*) : p \in \mathbb{C}[x,y]\}$ (closed already). $T \leftrightarrow A \xleftrightarrow{\Phi} id_{\{\lambda_1, \dots, \lambda_n\}}$.

$\sqrt{T} \leftrightarrow \sqrt{A} \xleftrightarrow{\Phi} (z \mapsto \sqrt{z}$ on $\{\lambda_1, \dots, \lambda_n\})$ only makes sense if $\lambda_i \geq 0$.

Say we want to make sense of e^T. As matrices: $e^A = \sum_{n\geq 0} A^n/n! = \sum_{n\geq 0} (UDU^*)^n/n! = U \sum_{n\geq 0} D^n/n! \; U^* = Ue^D U^*$.

So, $\Phi(e^T) = e^z$ on $\sigma(T)$ $(z = \lambda_i \Rightarrow e^z = e^{\lambda_i})$. $\qquad e^D = \begin{pmatrix} e^{\lambda_1} & & & 0 \\ & e^{\lambda_2} & & \\ & & \ddots & \\ 0 & & & e^{\lambda_n} \end{pmatrix}$

What about T^{-1}? T^{-1} would correspond via Φ to $1/z$, which is well-defined on $\sigma(T)$ iff $0 \neq \lambda_1, \dots, \lambda_n$.

$\sqrt{T^*T}$ is the func. $\sqrt{z}$ on $\sigma(T^*T)$ via cont. func. calc. for T^*T.

<u>Remark:</u> e^T makes sense for any $T \in B(H)$: $e^T = I + 1/1!\, T + 1/2!\, T^2 + \cdots$ (show it converges).

<u>Properties:</u> • $e^0 = I$

• $e^{T^*} = (e^T)^*$

• $e^{T+S} = e^T e^S$ if $TS = ST$.

<u>Thm (Fuglede):</u> If $N^*N = NN^*$ and $TN = NT$, then $T^*N = NT^*$. $\rightsquigarrow N^*T = TN^*$

<u>Putnam's Proof:</u> $TN = TN \Leftrightarrow Te^{\lambda N} = e^{\lambda N} T \;\; \forall \lambda \in \mathbb{C}$. To show $TN^* = N^*T \Leftrightarrow Te^{\lambda N^*} = e^{\lambda N^*} T \;\; \forall \lambda \in \mathbb{C}$.

So, want: $Te^{\lambda N^*} = e^{\lambda N^*} T \Leftrightarrow e^{-\lambda N^*} T e^{\lambda N^*} = T$. $f(\lambda) = e^{-\lambda N^*} T e^{\lambda N^*} = e^{-\lambda N^*} e^{\bar{\lambda} N} T e^{-\bar{\lambda} N} e^{\lambda N^*} = e^{-\lambda N^* + \bar{\lambda} N} T e^{-\bar{\lambda} N + \lambda N^*} = U(\lambda) T U(\lambda)^*$.

$U(\lambda) U(\lambda)^* = I = U(\lambda)^* U(\lambda)$, so $U(\lambda)$ unitary. So, $\|f(z)\| \leq 1 \cdot \|T\| \cdot 1 \Rightarrow f$ bounded on $\mathbb{C} \Rightarrow f$ constant (by Liouville).

$f(\lambda) = f(0) = T$.