Examples of $*$-alg, C^*-alg, & vN alg

(I) $B(H)$ $(M_n(\mathbb{C}))$ vN

(II) $L^\infty(X,\mu)$ vN if $\mu(X)<\infty$

$C(X)$ X compact unital C^*-alg

(III) G (discrete) countable group:

$\mathbb{C}[G]$, $C^*(G)$, $L(G) \subset B(\ell^2(G))$

Homework: 1. $B(H)$ w/ H separable (inf. dim.) has no trace ($\neq 0$). Also could use $T=[-,-]+[-,-]$...

Tao's blog

2. $DX-XD=I$ in $B(H)$? In finite dim, use trace.

$DX-XD \overset{\varepsilon}{\sim} I \Rightarrow \|D\|\|X\| > \text{big}$ (essentially, they can't be bounded)

$\lambda: G \to B(\ell^2(G))$ w/ ONB $\{\delta_h\}_{h\in G}$, $\lambda(g)\delta_h=\delta_{gh}$. $\rho(g)\delta_h=\delta_{hg^{-1}}$. $\forall g,h\in G$ Could do $\pi(g)\delta_h=\delta_{ghg^{-1}}=\rho(g)\lambda(g)\delta_h$.

Note that $[\lambda(g),\rho(g')]=0$, i.e. $\lambda(g)\rho(g')=\rho(g')\lambda(g)$ $\forall g,g'\in G$.

$L(G)=\overline{\operatorname{span}_{\mathbb{C}}\{\lambda(g):g\in G\}}^{so} \longrightarrow [L(G),R(G)]=0.$

$R(G)=\overline{\operatorname{span}_{\mathbb{C}}\{\rho(g):g\in G\}}^{so}$

s.o. closed

Indeed, $\lambda(g)$ commutes with all $\rho(h)$, so $\rho(h)\in\lambda(g)'$ $\forall g,h\in G$. $\Rightarrow \rho(h)\in(\operatorname{span}_{g\in G}\lambda(g))' \Rightarrow \operatorname{span}_{h\in G}\rho(h)\subset(\operatorname{span}_{g\in G}\lambda(g))'$.

* Algebra and analysis conversation

$\Rightarrow R(G)\subset(\operatorname{span}_{g\in G}\lambda(g))' \Rightarrow R(G)$ commutes with all $\lambda(g)$. So, $R(G)$, $L(G)$ commute.

Last time: $G=\mathbb{Z}_n \Rightarrow \mathbb{C}[G]=C^*(G)=L(G)=$ circulant matrices $\subset B(\ell^2(G)) \cong M_n(\mathbb{C})$. ($\cong\mathbb{C}^n$)

Note: if $|G|=n$ is Abelian, $C^*(G)=\mathbb{C}[G]\cong\mathbb{C}^n$.

$G=\mathbb{Z}$: $\ell^2(\mathbb{Z})$ has basis $\{\delta_n\}_{n\in\mathbb{Z}}$. Let $T=\lambda(1)$, so $T\delta_n=\delta_{n+1}$. Then $\lambda(\mathbb{Z})=\{T^k:k\in\mathbb{Z}\}$. What about closures?

Review

$C^*(\mathbb{Z}) = \overline{\text{span}\{T^n : n\in\mathbb{Z}\}}^{\|\ \|} = \overline{\text{span}\{T^n, (T^*)^n : n\in\mathbb{Z}\}}^{\|\ \|} = C^*(I,T) \cong C(\sigma(T)) = C(\mathbb{T})$.

To better understand $C^*(\mathbb{Z}), L(\mathbb{Z})$, it is useful to consider: $\ell^2(\mathbb{Z}) \overset{\sim}{=} L^2(\mathbb{T})$.

$(c_n)_{n\in\mathbb{Z}} \xleftarrow{F} f(z)$ $\leftarrow$ $f(z) = \sum_{n\in\mathbb{Z}} c_n z^n$, Fourier transform

So:

$$\begin{tikzcd}
\ell^2(\mathbb{Z}) \arrow[d, "T=\lambda(1)"] & L^2(\mathbb{T}) \arrow[l, "F"'] \arrow[d, "F^{-1}TF = M_z"] \\
\ell^2(\mathbb{Z}) \arrow[r, "F^{-1}"'] & L^2(\mathbb{T})
\end{tikzcd}$$

$L^2(\mathbb{T})$ has ONB $\{z^n : n\in\mathbb{Z}\}$. $F^{-1}TF(z^n) = F^{-1}T\delta_n = F^{-1}\delta_{n+1} = z^{n+1}$. So, $F^{-1}TF = M_z$.

So, $C^*(I,T) \cong \overline{\text{span}\{M_{z^k \bar{z}^\ell} : k,\ell \geq 0\}}^{\|\ \|}$. In fact, $F^{-1}\underset{C^*(\mathbb{Z})}{C^*(I,T)}F = C^*(I,T)$.

$\cong C(\mathbb{T})$. (Stone-Weierstrass) So, $C^*(I,T) \cong C(\mathbb{T})$.

So, $F^{-1}L(\mathbb{Z})F = \overline{C^*(I,M_z)}^{SO} \cong L^\infty(\mathbb{T})$, because $\overline{C(\mathbb{T})}^{SO} = L^\infty(\mathbb{T})$ (in $B(L^2(\mathbb{T}))$).

$\in C(\mathbb{T})'$

Recall that L^∞ is a MASA of $B(L^2)$. If $S \in \overline{C(\mathbb{T})}^{SO}$, then S commutes w/ $C(\mathbb{T})$. Hence with $L^\infty(\mathbb{T})$, so $S \in (L^\infty)' = L^\infty$.

Raises the question: $L^\infty(\mathbb{T}) = C(X)$

Trace on $L(G)$

$\tau(x) = \langle x\delta_e, \delta_e\rangle \quad \forall x \in L(G)$.

Claim: τ is positive: $\tau(x^*x) \geq 0$ $\leftarrow$ τ is a state

faithful: $\tau(x^*x) = 0 \iff x = 0$ $\quad$ τ generalizes the integral

tracial: $\tau(xy) = \tau(yx) \quad \forall x,y$

normal: $\tau(I) = 1$

sot cont: $x_i \xrightarrow{sot} x \Rightarrow \tau(x_i) \to \tau(x)$.

When a vN alg M has such a trace τ (so cont, faithful, positive), we say M is tracial.

Proof: $\tau(\lambda(g)) = \langle \lambda(g)\delta_e, \delta_e\rangle = \langle \delta_g, \delta_e\rangle = \begin{cases} 1 & \text{if } g=e \\ 0 & \text{if } g\neq e \end{cases}$. Similar to $\int_{\mathbb{T}} \sum_{n\in\mathbb{Z}} c_n z^n = c_0$.

positive: $\tau(x^*x) = \langle x^*x\delta_e, \delta_e\rangle = \langle x\delta_e, x\delta_e\rangle = \|x\delta_e\|^2 \geq 0$. faithful: $\tau(x^*x) = 0 \Rightarrow \|x\delta_e\|^2 = 0 \Rightarrow x\delta_e = 0$. We must show δ_e is separating for $L(G)$, meaning if

$x\delta_e=0$, $x\in L(G)$, then $x=0$. $x\in L(G)$, so x commutes with $R(G)$. $x\delta_e=0 \Rightarrow \rho(g)x\delta_e=0 \Rightarrow x\rho(g)\delta_e=0 \Rightarrow x\delta_{g^{-1}}=0$ $\forall g\in G \Rightarrow x=0$.

<u>Remark</u>: $L(G)\ni x\longmapsto x\delta_e\in \ell^2(G)$ is 1-1. If $x=\lambda(g)$, then $\lambda(g)\delta_e=\delta_g$. Call $u_g=\lambda(g)$; these generate $L(G)$ as a vN alg. Each $u_g\in L(G)$ corresponds to $\delta_g\in\ell^2(G)$. $\sum_{\text{finite},\, c_g\in\mathbb{C}} c_g u_g \to \sum c_g\delta_g$.

Conversely, since any $x\delta_e=\sum_{g\in G}c_g\delta_g$, we write $x=\sum c_g u_g$.