G group (countable). $u_g = \lambda(g) \in B(\ell^2(G))$.

$L(G) = \overline{\{\sum_{\text{finite}} c_g u_g : c_g \in \mathbb{C}\}}^{s.o.}$

If $G = \mathbb{Z}$: $\ell^2(\mathbb{Z}) \overset{F}{\cong} L^2(\mathbb{T})$

$\overline{\mathbb{C}[\mathbb{Z}]}^{so} = L(\mathbb{Z}) \cong L^\infty(\mathbb{T}) \subset B(L^2(\mathbb{T}))$

$u_n \longmapsto z^n \longmapsto M_{z^n} \in B(L^2(\mathbb{T}))$

$\overline{\mathbb{C}[\mathbb{Z}]}^{\|\|} = C^*(\mathbb{Z}) \cong C(\mathbb{T})$

$L(G)$ has trace τ: $\tau(x) = \langle x\delta_e, \delta_e \rangle$ $\forall x \in L(G)$ (so $\tau(u_g) = 0$ if $g \neq e$). So, $\tau(\sum_{fin} c_g u_g) = c_e$.

In the case $G = \mathbb{Z}$: $\tau(\sum_{fin} c_n z^n) = c_0$. This generalizes the integral; τ is like a noncommutative integral.

Why is $\tau(xy) = \tau(yx)$ $\forall x, y \in L(G)$?

If $x = u_g$ and $y = u_h$, then $\tau(u_{gh}) = \tau(u_{hg})$ is true because $gh = e$ iff $hg = e$.

So, for $x, y \in \mathbb{C}[G]$ (by linearity of τ), $\tau(xy) = \tau(yx)$.

Think $x_i \xrightarrow{so} 0$, $y_i \xrightarrow{so} 0$: $x_i 1 \to 0$, $y_i 1 \to 0$, but $x_i y_i 1 \to 0$? Not necessarily! However, $x_i y \xrightarrow{so} 0$ and $y x_i \xrightarrow{so} 0$

For $x, y \in L(G)$, take $x_i \xrightarrow{so} x$, $y_i \xrightarrow{so} y$ w/ $x_i, y_i \in \mathbb{C}[G]$. Does $x_i y_i \xrightarrow{so} xy$? Not necessarily.

* Operator algebraists in Romania

$\tau(x y_j) = \lim_i \tau(x_i y_j) = \lim_i \tau(y_j x_i) = \tau(y_j x)$ $\forall j$. Take limit wrt j to get $\tau(xy) = \tau(yx)$.

$(L(G), \tau)$ is a tracial vN algebra.

How can we work with elements of $L(G)$?

Recall that $L(G)$ embeds linearly in $\ell^2(G)$ via $x \longmapsto x\delta_e$. For $G = \mathbb{Z}$, $f \in L^\infty(\mathbb{T})$ embeds as $f \in L^2(\mathbb{T})$. So, $u_g \longmapsto \delta_g$ and

$\sum_{fin} c_g u_g \longmapsto \sum_{fin} c_g \delta_g$.

Note: $\overline{L(G)\delta_e}^{\|\|_2} = \ell^2(G)$.

If $x \in L(G)$, $x\delta_e \in \ell^2(G)$, so $x\delta_e = \sum_{g \in G} c_g \delta_g$. x is uniquely determined by the c_g's, and we write $x = \sum_{g \in G} c_g u_g$.

Note: $x \to x\delta_e$ is s.o. cont. So, sum is s.o. limit of partial sums.

If $f(x)=x\delta_e$, then f^{-1}(subset of $\ell^2(G)$) $= L(G)$. $\sum c_g\delta_g \xrightarrow{f^{-1}} \sum c_g u_g$.

$\subset B(\ell^2(\mathbb{Z}))$ $\quad \subset B(L^2(\mathbb{T}))$

For $G=\mathbb{Z}$: $L(\mathbb{Z}) \cong L^\infty(\mathbb{T}) \subset B(L^2(\mathbb{T}))$

$$u_n \longmapsto z^n \longmapsto M_{z^n}$$

$\Leftrightarrow \{M_z\}''$ $\quad \in L^2(\mathbb{T})$

So, $L^\infty(\mathbb{T})1 = L^\infty(\mathbb{T})$.

$L(\mathbb{Z}) = L^\infty(\mathbb{T}) \cong A = \{M_f : f\in L^\infty\} \subset B(L^2(\mathbb{T}))$

$$A \ni M_f \longmapsto M_f(1) = f$$

$$L^\infty \hookrightarrow L^2$$

So, any $f\in L^\infty$ is $f=\sum_{n\in\mathbb{Z}} c_n z^n$. We know $\sum|c_n|^2<\infty$, but that just makes $f\in L^2$. What else must c_n satisfy to guarantee $f\in L^\infty$? It's a mystery!

Let $\sum_{g\in G} c_g u_g$ be in $L(G)$. Can we say anything about the c_g's?

any vector in $\ell^2(G)$ $\qquad$ here g. say $gk=h$, then $(g,k)\mapsto(g,h)$ w/ $k=g^{-1}h$

$$\Big(\sum_{g\in G} c_g u_g\Big)\Big(\sum_h d_h\delta_h\Big) = \sum_{g,k\in G} c_g d_k u_g(\delta_k) = \sum_{g,k\in G} c_g d_k \delta_{gk} = \sum_{h\in G}\Big(\underbrace{\sum_{g\in G} c_g d_{g^{-1}h}}_{(c*d)_h}\Big)\delta_h.$$

$\in L(G)\subset B(\ell^2(G))$ $\qquad$ $d\in\ell^2(G)\longmapsto (c*d)\in\ell^2(G)$

So, $\sum c_g u_g$ takes any $\sum d_h\delta_h$ to $\sum(c*d)_h\delta_h$.

Thm: Let $(c_g)_{g\in G}$ be s.t. $\sum|c_g|^2<\infty$. Then $\sum c_g u_g\in L(G)$ iff the map $d\mapsto c*d$ is bounded as an operator on $\ell^2(G)$. $\quad$ * See proof in the book

of trivial center: $Z(L(G)) = \mathbb{C}\cdot I$

When is $L(G)$ a factor?

$\sum_h c_{hk^{-1}} u_h$ $\qquad$ $\sum_h c_{k^{-1}h} u_h$ $\qquad$ $hk^{-1}=g \Rightarrow h=gk$

Take $x\in Z(L(G))$, $x=\sum c_g u_g$. x commutes with u_k, $k\in G$. $xu_k=u_kx \Leftrightarrow \sum_g c_g u_{gk} = \sum_g c_g u_{kg}$ $\forall k$. $\Leftrightarrow c_{hk^{-1}} = c_{k^{-1}h}$ $\forall h,k$.

$\Leftrightarrow c_g = c_{k^{-1}gk}$. $\quad$ * Infinite conjugacy class groups