Thm: If G is a (countable) ICC group, then $L(G)$ is a factor (i.e. center is $\mathbb{C}I$).

Proof: A general element of $x\in L(G)$ is of the form $x=\sum_{g\in G} c_g u_g$ w/ $u_g=\lambda(g)$, $c_g\in\mathbb{C}$, $\sum|c_g|^2<\infty$ (and other conditions).

If $x\in Z(L(G))$, then $xu_h=u_hx$ $\forall h\in G$, so $\sum_g c_g u_{gh}=\sum_g c_g u_{hg}$. Left side: change variable $gh\mapsto g$ (so $g\mapsto gh^{-1}$).

Right side: $hg\mapsto g$ (so $g\mapsto h^{-1}g$). So, $\sum_g c_{gh^{-1}}u_g=\sum_g c_{h^{-1}g}u_g \Rightarrow c_{gh^{-1}}=c_{h^{-1}g}$ $\forall g\in G$. Call $h^{-1}g=k$, so $g=hk$. Then $c_{hkh^{-1}}=c_k$ $\forall k,h\in G$.

So, c is constant on $\{hkh^{-1}: h\in G\}$ = conjugacy class of k, $\forall k\in G$.

$\subset$ highly non-Abelian

An ICC (infinite conjugacy class) group is a group G such that all conj. classes (except the one of e) are infinite.

So, if G is ICC, c must be 0 on all those conj. classes. $\Rightarrow c_g=0$ $\forall g\neq e$. $\Rightarrow x=\sum c_g u_g=c_e u_e\in\mathbb{C}I$.

Examples of ICC Groups: • $G=SL(n,\mathbb{Z})=\{A\in M_n(\mathbb{Z}): \det(A)=1\}$, $n\geq 3$

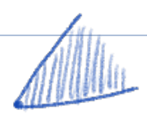
• $G=\bigcup_{n\geq 1} S_n=\{\sigma:\mathbb{N}\to\mathbb{N}$ permutation: $\sigma(i)=i$ $\forall i$ large$\}$.

• $G=\mathbb{F}_n$ $(n\geq 2)$ free group

Also $\langle T\xi,\xi\rangle\geq 0$

Recall: $T\in B(H)$ is called positive $(T\geq 0)$ if $T=T^*$ and $\sigma(T)\subset[0,\infty)$.

Equivalently (although not easy to see) $T\geq 0$ iff $T=X^*X$ for $X\in B(H)$.



$B(H)_+$ = cone of positive operators is closed to $+$ and mult. by positive scalars.

(1) If $T\geq 0$, then $T\leq\|T\|\cdot 1$ (meaning $\|T\|\cdot 1-T\geq 0$). $\begin{pmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{pmatrix}\leq\begin{pmatrix}\lambda & & 0\\ & \ddots & \\ 0 & & \lambda\end{pmatrix}$ w/ $\lambda=\max(\lambda_i)$.

like unitary diagonalization

Indeed, via Cont. Fct. Calculus, we identify T with $f(z)=z$, $z\in\sigma(T)$. For f, it is clear that $\overset{z}{\overset{\shortparallel}{f(z)}}\leq\overset{\sup\sigma(T)}{\overset{\shortparallel}{\|f(z)\|}}$

(2) If $T\leq S$ and $y\in B(H)$, then $yTy^*\leq ySy^*$. Can't really use CFC, unless y commutes

Indeed, we know $S-T\geq 0$ and want $y(S-T)y^*\geq 0$. $S-T=X^*X$ for some $X\in B(H)$, so $y(S-T)y^*=yX^*Xy^*=Z^*Z$ w/ $Z=Xy^*$.

The Gelfand-Naimark-Segal (GNS) Construction

$\forall \lambda \in \mathbb{C},\ a,b \in A$

Start with any unital algebra A over $\mathbb{C}$ with an involution $*$ $(A \ni a \mapsto a^* \in A$ s.t. $(a+b)^* = a^* + b^*$, $(a \cdot b)^* = b^* \cdot a^*$, $(\lambda \cdot a)^* = \bar{\lambda} a^*)$

and a faithful state $\varphi: A \to \mathbb{C}$ $(\varphi$ is linear, $\varphi(x^*x) \geq 0$ w/ $=$ iff $x=0$, $\varphi(1)=1)$.

Examples: (I) $A = C([0,1])$ w/ $f^* = \bar{f}$ and $\varphi(f) = \int_{[0,1]} f$ $\forall f$.

(II) $A = \bigcup_{n \geq 1} M_{2^n}(\mathbb{C})$ w/ $x^* = \bar{x}^T$ and $\varphi|_{M_{2^n}} = \tau|_{M_{2^n}}$ $(\tau = 1/2^n \cdot \mathrm{Tr})$ $\forall x$.

Here $M_2(\mathbb{C}) \hookrightarrow M_4(\mathbb{C})$ via $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Similarly, $M_4(\mathbb{C}) \hookrightarrow M_8(\mathbb{C})$ via $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$.

The GNS Construction:

For $A = C([0,1])$ w/ $\varphi = \int$, $H = L^2([0,1])$.

From (A, φ) we construct a Hilbert space H and an injective $*$-morphism $\pi: A \to B(H)$ as follows:

Define $\langle x,y \rangle_\varphi = \varphi(y^*x)$ $\forall x,y \in A$. In particular, $\|x\|_{2,\varphi} = \langle x,x \rangle_\varphi^{1/2} = \varphi(x^*x)^{1/2}$. Define $H = \overline{A}^{\|\ \|_{2,\varphi}}$, this is a Hilbert space.

So, A embeds in H. To avoid confusion, we write $\hat{a}$ when we think of $a \in A$ as an element of H. So, $H = \overline{\{\hat{a} : a \in A\}}^{\|\ \|_{2,\varphi}}$.

$\pi: A \to B(H)$ is defined by $\pi(a)(\hat{b}) = \widehat{ab}$ $\forall a,b \in A$. We must show that π extends to all H.

So, if $b_n \to x \in H$ and $b_n' \to x \in H$ w/ $b_n, b_n' \in A$, WTS: $\lim \overset{\widehat{ab_n}}{\overset{\shortparallel}{\pi(a)\hat{b}_n}} = \lim \overset{\widehat{ab_n'}}{\overset{\shortparallel}{\pi(a)\hat{b}_n'}}$ (and limits exist).

1st: $\|\overset{\widehat{ab}}{\overset{\shortparallel}{\pi(a)\hat{b}}}\|_{2,\varphi} = \|\widehat{ab}\|_{2,\varphi} = \varphi(b^*a^*ab)^{1/2} \leq \varphi(b^*\|a^*a\|b)^{1/2}$. Will finish next time!