(A *-alg, φ) ⟼ $H = \overline{A}^{\|\cdot\|_{2,\varphi}}$. $\pi: A \to B(H)$??? A acts on H, how to extend to H? Need norm with properties.

Def: A pre-C^* algebra is any *-algebra A with a unit 1 and a norm $\|\ \|$ satisfying $\|xy\| \le \|x\|\|y\|$ and $\|x^*x\| = \|x\|^2$ $\forall x, y \in A$.

Def: An abstract C^*-algebra is a pre-C^* algebra which is complete in $\|\ \|$ (or Banach algebra).

Def: A concrete (unital) C^*-algebra is any Banach unital *-algebra of $B(H)$.

* can show φ always exists (like GNS below)

Thm: (Gelfand) Any abstract C^*-alg. is a concrete C^*-alg. (on some $B(H)$).

So, abstract = concrete. Note: once you have $A \subset B(H)$, you can do $\overline{A}^{s.o.}$.

Two examples to keep in mind:

(1) $A = C(C_0(1))$ with $\|\ \|_\infty$ is an abstract C^*-alg.

(2) $D = \bigcup_{k\ge 1} M_{2^k}(\mathbb{C})$ via $x \hookrightarrow \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$ with matrix $\|\ \|$ (note $\|x\| = \left\|\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}\right\|$ $\forall x \in M_{2^k}(\mathbb{C})$) is a pre-$C^*$-algebra.

$\overline{D}^{\|\ \|}$ (completion of D in $\|\ \|$) is an abstract C^*-algebra.

The Gelfand-Naimark-Segal (GNS) Construction

Start with A a (unital) pre-C^*-algebra and $\varphi: A \to \mathbb{C}$ a linear, positive, faithful, norm-cont. map w/ $\varphi(1) = 1$.

Then there is a unique map $\pi: A \to B(H)$, where $H = \overline{A}^{\|\cdot\|_{2,\varphi}}$ such that π is a *-alg. morphism, π is injective, norm-preserving, and $\pi(a)\hat{b} = \widehat{ab}$ $\forall a, b \in A$ (where $\hat{b}$ denotes b via the embedding $A \hookrightarrow \overline{A}^{\|\cdot\|_{2,\varphi}}$).

$A \hookrightarrow H = \overline{A}^{\|\cdot\|_{2,\varphi}}$

*-alg

φ

$A \xrightarrow{\pi} B(H)$, $\pi(a)\hat{b} = \widehat{ab}$; $\pi(a)$ is defined on a, but need $\overline{A}^{\|\cdot\|_{2,\varphi}}$.

Can't extend it unless we know some cont. prop. for $\pi: a_n \to a \Rightarrow \pi(a_n) \to \pi(a)$?

$\overline{C(\Omega)}^{\|\cdot\|_{2,\varphi}} = L^2(\Omega)$. $C(\Omega) \subset B(L^2(\Omega))$

$f \longmapsto \hat{f}$ $\qquad$ $f \longmapsto M_f(g) = \widehat{fg}$

Proof: We may assume A is closed in $\|\cdot\|$ (if not, replace A by its completion in $\|\cdot\|$ and extend φ by cont.).

Recall on A we defined $\langle x,y\rangle = \varphi(y^*x)$ $\forall x,y \in A$ and $\|x\|_{2,\varphi} = \varphi(x^*x)^{1/2}$. Let H = completion of the pre-Hilbert space $\bigoplus_{a\in A}$ $(A, \langle\cdot,\cdot\rangle) = \overline{A}^{\|\cdot\|_{2,\varphi}}$. Let $\pi(a)\hat{b} = \widehat{ab}$ $\forall a,b \in A$ (so $\pi(a)$ is a linear map on $A \subset H$). We want to extend $\pi(a)$ to H: for $\hat{f} = \lim \hat{b}_n$,

$\pi(a)\hat{f} = \lim_{n\to\infty} \pi(a)\hat{b}_n = \lim_{n\to\infty} \widehat{ab_n}$. To show this is well-defined, we must show: if $\hat{b}_n \to \hat{f}$, $\hat{c}_n \to \hat{f}$ w/ $b_n, c_n \in A$, then $\lim \widehat{ab_n} = \lim \widehat{ac_n}$. Need: if $\hat{b}_n - \hat{c}_n \to 0$, then $\widehat{a(b_n - c_n)} \to 0$. Note: $\|\widehat{ab}\|_2^2 = \varphi(b^*a^*ab) \le \varphi(b^*\|a\|^2 1 b) = \|a\|^2\varphi(b^*b) = \|a\|^2\|b\|_2^2$ (positive; as operator; in A).

So, indeed if $\hat{b}_n - \hat{c}_n \to 0$, then $\widehat{a(b_n - c_n)} \to 0$. So, $\pi(a)$ is well-defined on H, and it is in $B(H)$ as $\|\pi(a)\| \le \|a\|$.

Remark: In an abstract C^*-alg A, we can define $A_+ = \{x^*x : x \in A\}$ and prove properties like: $a^*a \le \|a^*a\| 1$, $x \le y \Rightarrow b^*xb \le b^*yb$.

Check: π is a $*$-morphism and $\|\pi(a)\| = \|a\|$ (so π is injective).

Example 1: $A = C(\Omega)$, $\varphi(f) = \int_0^1 f\,dm$. $H = \overline{A}^{\|\cdot\|_{2,\varphi}} = L^2(\Omega, m)$, $\|f\|_{2,\varphi} = \varphi(f^*f)^{1/2} = (\int |f|^2)^{1/2} = \|f\|_2$.

$\pi(f)(\hat{g}) = \widehat{fg}$ $\forall f,g \in A$, so $\pi(f) = M_f$ = left mult. by f and it extends to $L^2(\Omega, m)$. So, $\pi : C(\Omega) \to B(L^2(\Omega))$ with $\|\cdot\|_\infty$, $f \longmapsto M_f$.

Remark: If $A \subset B(H)$, we can consider $\overline{A}^{S.O.}$, which is a v.N. alg. (since A was $*$-closed, unital).

$\overline{C(\Omega)}^{\|\cdot\|_{B(L^2)}} = L^2(\Omega)$.

Example 2: $\Omega = \bigcup_{n\ge1} M_{2^n}(\mathbb{C})$ yields $\overline{\mathcal{B}}^{\|\cdot\|}$ C^*-alg. (different), $\overline{\mathcal{B}}^{S.O.}$ v.N. alg. (same).

Example 2': $\mathcal{C} = \bigcup_{n\ge1} M_{3^n}(\mathbb{C})$, yields $\overline{\mathcal{C}}^{\|\cdot\|}$ C^*-alg., $\overline{\mathcal{C}}^{S.O.}$ v.N. alg.

$x \hookrightarrow \begin{pmatrix} x & & \\ & x & \\ & & x \end{pmatrix}$

Thm: $\overline{\bigcup M_{2^n}}^{\|\cdot\|} \neq \overline{\bigcup M_{3^n}}^{\|\cdot\|}$ (as C^*-algebras, so not a (norm-preserving) $*$-isomorphism).

First, why $U_{M_{2k}} \cong U_{M_{3k}}$ as *-algebras? * 1st K theory, K, invariant

$\tau(1)=1$ ↓

Let's look, for any A an alg. with a trace τ, at $\{\tau(x):x$ proj. in $A\}$.

On the left, trace should involve powers of 2, whereas on the right powers of 3.