$M_2(\mathbb{C}) \subset M_4(\mathbb{C})$, $x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$.

tensor or Kronecker product

A better way: $M_4(\mathbb{C}) = M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$, and $M_2(\mathbb{C})$ embeds in it as $M_2(\mathbb{C}) \otimes I$.

$$\underbrace{A}_{\in M_2(\mathbb{C})} \otimes B := \begin{pmatrix} Ab_{11} & Ab_{12} \\ Ab_{21} & Ab_{22} \end{pmatrix}.$$

So, $A \otimes I = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}$. So, $M_2 \subset M_4 \subset M_8 \subset \cdots$ means $M_2 \subset M_2 \otimes M_2 \subset M_2 \otimes M_2 \otimes M_2 \subset \cdots$, where each embedding is done on the 1st part of the tensor.

So, $\bigcup_{n \geq 1} M_{2^n}(\mathbb{C})$ can be denoted as $\bigotimes_{n \geq 1} M_2(\mathbb{C})$. $\overline{\bigcup_{n \geq 1} M_{2^n}(\mathbb{C})}^{\|\ \|}$ is denoted $\overline{\bigotimes}^{\|\ \|} M_2(\mathbb{C})$, and $\overline{\bigcup_{n \geq 1} M_{2^n}(\mathbb{C})}^{s.o.}$ is denoted $\overline{\bigotimes}^{s.o.}_{n \geq 1} M_2(\mathbb{C})$ or $\overline{\bigotimes} M_2(\mathbb{C})$.

<u>Thm. 1</u>: $\overline{\bigotimes}^{\|\ \|} M_2(\mathbb{C}) \neq \overline{\bigotimes}^{\|\ \|} M_3(\mathbb{C})$. More generally, $\overline{M_{k_1}(\mathbb{C}) \otimes M_{k_2}(\mathbb{C}) \otimes \cdots}^{\|\ \|} \neq \overline{M_{\ell_1}(\mathbb{C}) \otimes M_{\ell_2}(\mathbb{C}) \otimes \cdots}^{\|\ \|}$ if $\{k_1, k_2, \ldots\}$ and $\{\ell_1, \ell_2, \ldots\}$ don't have same prime factors.

These are called AFD C^*-algebras (approximately finite dimensional).

<u>Thm. 2</u>: $\overline{\bigotimes}^{s.o.} M_2(\mathbb{C}) = \overline{\bigotimes}^{s.o.} M_3(\mathbb{C}) = \overline{M_{k_1} \otimes M_{k_2} \otimes \cdots}^{s.o.}$ $\forall k_1, k_2, \ldots \geq 2$.

Simplest infinite dimensional nonAbelian v.N. alg. (compare to ℓ^∞, $M_n(\mathbb{C})$, or $L(G)$)

This v.N. alg. is denoted by $\mathcal{R}$ and called the hyperfinite type II_1 factor.

<u>Plan of proof for Thm. 1</u>:

$A = \overline{\bigotimes}^{\|\ \|} M_2(\mathbb{C})$, $B = \overline{\bigotimes}^{\|\ \|} M_3(\mathbb{C})$. Assume that $\Phi: A \to B$ isomorphism exists (preserves $+, \cdot, {}^*, \underbrace{\|\ \|}_{\text{automatic}}$).

A has a unique normalized trace τ, meaning: $\tau(1) = 1$, $\tau: A \to \mathbb{C}$ linear, pos., $\tau(xy) = \tau(yx)$ $\forall x, y \in A$, cont. Indeed, $\tau|_{M_{2^n}(\mathbb{C})} = \underbrace{1/2^n \, Tr}_{\text{unique matrix trace (up to scalar mult)}}$.

Similarly, B has a unique trace τ' with $\tau'(1) = 1$. $\begin{tikzcd} A \arrow[r, "\Phi"] \arrow[d, "\tau"'] & B \arrow[d, "\tau'"] \\ \mathbb{C} & \mathbb{C} \end{tikzcd}$ $\Rightarrow \tau' \circ \Phi = $ trace on $A = \tau$.

If $p \in \mathcal{P}(A)$ (proj. of A), $p = p^2 = p^*$ $\Rightarrow \Phi(p) = \Phi(p)^2 = \Phi(p)^*$ $\Rightarrow \Phi(p) \in \mathcal{P}(B)$. Also, $\tau(p) = \tau' \circ \Phi(p) = \tau'(\Phi(p))$.

So, $\tau(P(A)) = \tau'(P(B))$ (as sets). If we show $\underset{K_0(A)}{\overset{\shortparallel}{\tau(P(A))}} = \{k/2^n : 0 \le k \le 2^n,\ n \ge 0\}$ and $\underset{K_0(B)}{\overset{\shortparallel}{\tau'(P(B))}} = \{\ell/3^m : 0 \le \ell \le 3^m,\ m \ge 0\}$, then we have a contradiction. We prove $\tau(P(A)) = \{k/2^n : 0 \le k \le 2^n,\ n \ge 0\}$.

"$\supseteq$" is clear: $\begin{pmatrix} \underbrace{1 \ddots 1}_{k} & 0 \\ 0 & \underbrace{0 \ddots 0}_{2^n-k} \end{pmatrix} \in M_{2^n}(\mathbb{C})$ is a proj. of trace $k/2^n$. * In $M_n(\mathbb{C})$, any proj. is unitarily diagonalizable to something of this form.

"$\subseteq$" is the main part of the proof. Start with p = proj. in $A = \overline{\bigcup M_{2^n}(\mathbb{C})}^{\|\,\|}$.

<u>Steps of proof:</u>

(1) For $\varepsilon > 0$ (small as we need), there is a proj. $q \in M_{2^n}(\mathbb{C})$ for some n with $\|p-q\| < \varepsilon$.

(2) We show that if p, q are proj. in A and $\|p-q\| < \varepsilon$ (small enough), then $\tau(p) = \tau(q)$. * Compare to $p, q \in M_n(\mathbb{C})$

<u>Lemma</u>: If A is a unital C^*-alg. and $p, q \in P(A)$ with $\|p-q\| < 1$, then $q = UpU^*$ for some unitary $U \in A$.

So, $\tau(q) = \tau(UpU^*) = \tau(pU^*U) = \tau(p)$.

Think of $B(H)$: if P_k is 3-dim. and P_ℓ is 4-dim., then $\|P_k - P_\ell\| \ge 1$. $\|P_k - P_\ell\| < 1 \Rightarrow k = \ell \Rightarrow P_k, P_\ell$ unitarily equivalent.

Need $U \in A$ (not just $B(H)$) in order to use τ!

<u>Proof of lemma (hence of (2))</u>: We want a unitary U with $q = UpU^*$, so $Up = qU$.

First, can we think of any $x \neq 0$ in A in terms of p, q with $xp = qx$? $x = qp$ works! $(qp)p = q(qp)$. However, it's not going to be unitary, as qp (as an operator on $B(H)$) is 0 on $p(H)^\perp$. $(1-p)(H)$ $p(H) \xrightarrow{qp} q(H)$, $p(H)^\perp \to 0$. Note $((1-q)(1-p))p = q((1-q)(1-p))$. both 0. So, let $x = qp + (1-q)(1-p)$.

$$H \left\{ \begin{array}{l} p(H) \xrightarrow{x} qp(H) \subseteq q(H) \\ (1-p)(H) \xrightarrow{x} (1-q)(1-p)(H) \subseteq (1-q)(H) \end{array} \right.$$

So, $xp = qx$, but is x unitary? $x^*x = (pq + (1-p)(1-q))(qp + (1-q)(1-p)) = pqp + 0 + 0 + (1-p)(1-q)(1-p) = pqp + (1-p-q+pq)(1-p)$

$= pqp + (1-p) - q(1-p) + pq(1-p) = pqp + 1 - p - q + qp + pq - pqp = 1 - p - q + qp + pq = 1 - p^2 - q^2 + qp + pq = 1 - (p-q)^2$.

Note that x^*x is invertible, because $\|1 - x^*x\| = \|(p-q)^2\| \le \|p-q\|^2 < 1$ (<u>fact</u>: if $\|1-T\| < 1$, T has inverse $S = 1 + T + T^2 + T^3 + \cdots$).

Hence, $|x| = (x^*x)^{1/2}$ is invertible. Polar decomp: $x = \overset{\text{p.i.}}{\downarrow}{U}|x| \Rightarrow U = x|x|^{-1}$. U is unitary b/c U is p.i. of kernel $\ker|x| = 0$.

We show: $Up = qU$. Indeed, $xp = qx \Rightarrow px^* = qx^*$. $xp = qx \Rightarrow x^*xp = x^*qx = px^*x$, so x^*x commutes with p. Hence, $\sqrt{x^*x} = |x|$ commutes with p, as $\sqrt{x^*x} \in C^*(1, x^*x)$. Will continue next time!