Important Topologies on $B(H)$

$\| \|$ topology, s.o.t., w.o.t.

Def: The strong operator topology is the coarsest topology on $B(H)$ making all functions $B(H) \ni T \to T\xi \in H$ continuous $(\forall \xi \in H)$.

That is to say: a net $(T_i)_{i\in I}$ converges to T iff $T_i\xi \to T\xi$ $\forall \xi \in H$.

Alternatively, sot is the locally convex topology on $B(H)$ given by the seminorms $B(H) \ni T \to \|T\xi\| \in \mathbb{C}$ $(\forall \xi \in H)$

Furthermore, a sub-basis of this topology is given by $U(T, \xi, \varepsilon) = \{S \in B(H) : \|S\xi - T\xi\| < \varepsilon\}$ (indexed by $T \in B(H)$, $\xi \in H$, $\varepsilon > 0$).

So, a basis for sot is $V(T, \xi_1, \dots, \xi_n, \varepsilon) = \{S \in B(H) : \|S\xi_k - T\xi_k\| < \varepsilon \ \forall k = 1, \dots, n\}$ (indexed by $\xi_1, \dots, \xi_n \in H$, $\varepsilon > 0$).

unit ball

Remark: if H is inf. dim., sot is not metrizable (it is metrizable on $B(H)_1$ for H separable).

like convergence in coefficients

Similarly: the weak operator topology is given by $T_i \to T$ iff $\langle T_i\xi, \eta\rangle \to \langle T\xi, \eta\rangle$ $\forall \xi, \eta \in H$.

$\| \|$ convergence $\Rightarrow$ sot convergence $\Rightarrow$ wot convergence.

Banach-Alaoglu

Review

Thm: $B(H)_1$ is compact in wot.

$T \mapsto (\langle T\xi, \eta\rangle)_{\xi,\eta\in H}$ and Tychonoff

In general, $\| \|$, sot, wot are distinct topologies (unless $\dim H < \infty$).

Example to show wot $\neq$ sot on H w/ ONB $\{e_0, e_1, e_2, \dots\}$

Let $T_n \in B(H)$ taking $T_n(e_0) = e_n$ and $T_n(e_0^\perp) = 0$. Claim: $T_n \to 0$ wot, $T_n \not\to 0$ sot.

$T_n(e_0) = e_n \not\to 0$ (as $\|e_n\| = 1$). So, $T_n(e_0) \not\to 0 \Rightarrow T_n \overset{sot}{\not\to} 0$.

Why does $T_n \xrightarrow{wot} 0$? Let's check $\langle T_n x, y \rangle \to 0$ $\forall x, y \in H$. $x = \sum_{i \geq 0} c_i e_i$, $y = \sum_{j \geq 0} d_j e_j \Rightarrow \langle T_n x, y \rangle = \langle T_n c_0 e_0, y \rangle = \langle c_0 e_n, y \rangle$

$= \langle c_0 e_n, d_n e_n \rangle = c_0 \overline{d_n} \to 0$. ⊛ Look at book

<u>Why is $\|\;\| \neq sot$?</u>

$\overline{F(H)}^{\|\;\|} = K(H)$ $F(H)$ = finite rank operators

$\overline{F(H)}^{sot} = B(H)$

<u>Def</u>: Let $A \subset B(H)$, $A \neq \emptyset$.

(1) We say A is a $*$-algebra if A is an algebra over $\mathbb{C}$ and $*$-closed ($T \in A \Rightarrow T^* \in A$)

(2) We say A is a C^*-algebra if A is a $*$-algebra and $\overline{A}^{\|\;\|} = A$.

(3) We say A is a von Neumann algebra if A is a unital $*$-algebra and $\overline{A}^{sot} = A$.

<u>Remark</u>: Any vN alg. is C^*: if $T_n \xrightarrow{\|\;\|} T$ and $T_n \in A$, then $T_n \xrightarrow{sot} T$ so $T \in A$ (as A is vN).

If $\dim H$ is finite, then (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

<u>Remark</u>: $A = K(H)$ (H inf. dim.), $A = \overline{A}^{\|\;\|}$, but $\overline{A}^{sot} = B(H)$. So, $K(H)$ is C^* but not vN.

<u>Examples of vN algebras</u>

(I) $B(H)$ i.e. $H \cong \mathbb{C}^n$, $B(H) \cong M_n(\mathbb{C})$.

(II) <u>Thm</u>: If (X, μ) is a finite measure space, then $A = L^\infty(X, \mu) \subset B(L^2(X, \mu))$ is a vN alg.

First, let's talk about commutants of sets.

Def: $A \subset B(H)$, $A \neq \emptyset$. A' = commutant of $A = \{T \in B(H) : TS = ST \ \forall S \in A\}$.

Remark: A' is wot closed: if $T_i \to T$ w/ $T_i \in A'$, then we know $T_i S = S T_i$ $\forall S \in A$ and want $TS = ST$ for $TS\xi = ST\xi$ $\forall \xi \in H$.

$$\begin{array}{ccc} T_i S\xi & = & S T_i \xi \\ \downarrow & & \downarrow \\ TS\xi & = & ST\xi \end{array}$$

Remark: A' is an algebra, and $I \in A'$.

Remark: If A is $*$-closed, so is A'. ✪ Exercise

So, A' is a vN alg!

Later: M is a vN iff M''.

Properties of commutants

$A \subset B \subset B(H) \Rightarrow B' \subset A' \Rightarrow A'' \subset B''$.

Also: $A \subset A''$, but $A \neq A''$ in general.

$A \subset A'' \Rightarrow A''' \subset A'$ and $A' \subset (A')'' = A''' \Rightarrow A' = A'''$.

Remark: $A \subset B(H)$ and A is Abelian iff $A \subset A'$.

If in fact $A = A'$, this means: if $T \in B(H)$ is such that T commutes w/ all operators in A, then $T \in A$. In other words,

A is maximal Abelian (MASA).

Ex: D = diagonals in $M_n(\mathbb{C})$ is MASA (so $D = D'$).