$$\begin{tikzcd} & B(H) \arrow[dl] \arrow[d] \arrow[dr] & \\ \|\ \| & \text{sot} & \text{wot} \end{tikzcd}$$

metrizable

more natural

$B(H)_1$ compact

S' is sot closed

Thm: If (X,μ) is a finite measure space, then $A = L^\infty(X,\mu) \subset B(L^2(X,\mu))$ is a von Neumann algebra.

$f \mapsto M_f$

So, A is a MASA

Proof: We show $A = A'$. In particular, A is sot closed. Since A is Abelian, we know $A \subset A'$. We must show $A' \subset A$.

So, let $T \in A'$. $\Rightarrow TM_g = M_gT$ $\forall g \in L^\infty$. WTS: $T = M_f$ w/ $f \in L^\infty$ (if $T = M_f$, then $T(1) = M_f(1) = f$). Let $f = T(1) \in L^2(X,\mu)$.

We show: (i) $T = M_f$ and (ii) $f \in L^\infty$.

$TM_g(1) = M_gT(1) \Rightarrow T(g) = gf = fg = M_f(g)$ $\forall g \in L^\infty$.

Also, $\|T(g)\|_2 \le \|T\|\|g\|_2 \Rightarrow \|fg\|_2 \le \|T\|\|g\|_2$ $\forall g \in L^\infty$. Let $g = \chi_E$ ($E \subset X$ measurable). Then $\|f\chi_E\|_2^2 \le \|T\|^2\|\chi_E\|_2^2$

$\Rightarrow \int_E |f|^2 \le \|T\|^2\mu(E) = \int_E \|T\|^2$ (if $\int_E |f| \le \int_E |g|$ $\forall E$ measurable, then $|f| \le |g|$ a.e.) $\Rightarrow |f| \le \|T\|$ a.e. $\Rightarrow \|f\|_\infty \le \|T\| \Rightarrow f \in L^\infty$ and

$T = M_f$ on L^∞. Hence, $T = M_f$ on $\overline{L^\infty}^{\|\ \|_2} = L^2$.

(III) Let G be a (finite or countable) group. $\ell^2(G) = \{f : G \to \mathbb{C} : \sum_{g\in G} |f(g)|^2 < \infty\} = \{(\alpha_g)_{g\in G} : \sum_{g\in G} |\alpha_g|^2 < \infty\}$.

Basis: $(\delta_g)_{g\in G}$ w/ $\delta_g(h) = \begin{cases} 1 & \text{if } g = h \\ 0 & \text{if } g \ne h \end{cases}$. On $B(\ell^2(G))$, we define unitaries $u_g\delta_h = \delta_{gh}$ $\forall h \in G$.

Check: $u_g^* = u_{g^{-1}}$, $u_g \cdot u_{g^{-1}} = u_{g^{-1}} \cdot u_g = u_e = I$ ($e = \mathrm{id}_G$).

The Left Regular Representation of G

This is the map $\lambda : G \to B(\ell^2(G))$, $\lambda(g) = u_g$. This is an embedding of G into the unitaries of $B(\ell^2(G))$ satisfying

$\lambda(gh) = \lambda(g)\lambda(h)$ and $\lambda(g^{-1}) = \lambda(g)^*$ $\forall g,h \in G$.

Recall $G \hookrightarrow S_n$, $g : G \to G$, $h \mapsto gh$.

Indeed, $\lambda(gh)\delta_k = \lambda(g)\lambda(h)\delta_k \iff \delta_{ghk} = \delta_{g(hk)}$ $\forall k \in G$.

So, $G \cong \lambda(G)$ as groups. λ is denoted by λ_G.

Similarly, one can define a right regular representation $\rho : G \to B(\ell^2(G))$ w/ $\rho(g)\delta_h = \delta_{hg^{-1}}$. Do we have $\rho(g_1 g_2) = \rho(g_1)\rho(g_2)$?

$\rho(g_1 g_2)\delta_h = \rho(g_1)\rho(g_2)\delta_h \iff \delta_{h(g_1 g_2)^{-1}} = \delta_{h g_2^{-1} g_1^{-1}}$. Hence, $\rho(G) \cong G$.

Def: $\mathbb{C}[G] = \operatorname{span}\{\lambda(g) : g \in G\} \subset B(\ell^2(G))$.

$C^*(G) = \overline{\mathbb{C}[G]}^{\| \ \|}$.

$L(G) = vN(G) = \overline{\mathbb{C}[G]}^{SO}$.

$\mathbb{C}[G] \subset C^*(G) \subset L(G)$.

free groups

$L(\mathbb{F}_2) \cong L(\mathbb{F}_3)$? Big open problem, generated Voiculescu's Free Probability.

Example: $G = \mathbb{Z}_3 = \{0,1,2\}$. $\ell^2(G) = \operatorname{span}\{\delta_0, \delta_1, \delta_2\} \cong \mathbb{C}^3$. $\lambda(g)\delta_h = \delta_{g+h}$ $\forall g,h \in \mathbb{Z}_3$. $\lambda(g)$ can be identified to a 3×3 matrix, so

$T \longmapsto [T]_{\{\delta_0,\delta_1,\delta_2\}}$

$B(\ell^2(G)) \cong B(\mathbb{C}^3) \cong M_3(\mathbb{C})$.

$\lambda(0) = I$.

$\lambda(1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$.

$\lambda(2) = \lambda(1)\cdot\lambda(1) = \lambda(1)^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

$\mathbb{C}[\mathbb{Z}_3] = \left\{ aI + b\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} : a,b,c \in \mathbb{C} \right\} = \left\{ \begin{pmatrix} a & b & c \\ c & a & b \\ b & a & c \end{pmatrix} : a,b,c \in \mathbb{C} \right\} \subset M_3(\mathbb{C})$, the circulant matrices.

From here, we can see that these commute.

Example: $G = \mathbb{Z}$. $\mathbb{C}[G] = \operatorname{span}\{u_0, u_1, u_{-1}, u_2, u_{-2}, \dots\}$. $u_k = \lambda(k) = \lambda(\overbrace{1 + \cdots + 1}^{k \text{ times}}) = \lambda(1)^k = u_1^k$. If $u_1 = T$, $\mathbb{C}[G] = \operatorname{span}\{I, T, T^{-1}, T^2, T^{-2}, \dots\}$.

$T : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$, $T(\delta_n) = \delta_{n+1}$, the right shift on $\ell^2(\mathbb{Z})$. $\mathbb{C}[G] = \{P(T, T^*) : P \in \mathbb{C}[x,y]\}$.

unit circle

Recall: T normal $\Rightarrow C^*(I,T) \cong C(\sigma(T))$. So, $C^*(\mathbb{Z}) = C^*(I,T) \cong C(\sigma(T)) = C(\mathbb{T})$.

Remember, the Fourier transform gives $\ell^2(\mathbb{Z}) \longleftrightarrow L^2(\mathbb{T})$.

To better understand T and $C^*(T)$, $vN(T)$, first let's identify $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{T})$ via the Fourier transform.

$L^2([0,1], m) \longrightarrow \ell^2(\mathbb{Z})$ $\quad$ $t \in [0,1] \longmapsto z = e^{2\pi i t} \in \mathbb{T}$ $\quad$ $L^2(\mathbb{T}, m) \xrightarrow{F} \ell^2(\mathbb{Z})$ $\quad$ $F(\sum_{n\in\mathbb{Z}} c_n z^n) = (c_n)_{n\in\mathbb{Z}}$

$\sum c_n e^{2\pi i n t} \longmapsto (c_n)$ $\quad$ $\sum c_n z^n \longmapsto (c_n)$ $\quad$ Fourier transform

$$\begin{tikzcd}
L^2(\mathbb{T}) \arrow[r, shift left, "F"] \arrow[d, dashed] & \ell^2(\mathbb{Z}) \arrow[l, shift left, "F^{-1}"] \arrow[d, "T"] \\
L^2(\mathbb{T}) \arrow[r, shift left, "F"] & \ell^2(\mathbb{Z}) \arrow[l, shift left, "F^{-1}"]
\end{tikzcd}$$

$F^{-1}TF = M_z \rightsquigarrow$

$M_z \in B(L^2(\mathbb{T}))$, $\sigma(M_z) = \mathbb{T}$.