(Bounded)

## Borel Functional Calculus

$x$ is normal in $B(H)$, $\Phi : \mathcal{B}(\sigma(x)) \to \mathrm{v.N.}(x)$, $f \mapsto \Phi(f) = f(x)$ s.t.

(1) $\Phi$ extends the cont. func. calc. (on $\sigma(x)$)

(2) if $f_n \to f$ pointwise is a bounded sequence in $\mathcal{B}(\sigma(x))$, then $f_n(x) \xrightarrow{s.o.} f(x)$ in $B(H)$.

## Applications

**Thm:** $U \in B(H)$ is unitary if and only if $U = e^{iH}$ for $H$ Hermitian in $B(H)$ ($H = H^*$). Think $|z|=1 \Leftrightarrow z = e^{i\theta}$, $\theta = \bar{\theta}$.

Proof: ($\Leftarrow$) If $H = H^*$, then $UU^* = e^{iH}(e^{iH})^* = e^{iH}e^{-iH} = I$. $U^*U = I$ similarly.

($\Rightarrow$) If $U$ is unitary, we know $C^*(U) = C(\sigma(U))$, $U \leftrightarrow f(\lambda) = \lambda$, $I = U^*U \leftrightarrow \bar{\lambda}\lambda = 1$, so $|\lambda| = 1$. Hence, $\sigma(U) \subset \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Let $g(t) = e^{it}$, $g : [0, 2\pi) \to \mathbb{T}$ bijective. $g^{-1} : \mathbb{T} \to [0, 2\pi)$ is Borel but not cont! So, $g^{-1}$ is defined and Borel on $\sigma(U) \subset \mathbb{T}$.

$\Rightarrow H := g^{-1}(U) \in \mathrm{v.N.}(U) \subset B(H)$, and $H = H^*$ b/c $g^{-1} = \overline{g^{-1}}$ (so image is real) $\xRightarrow{\text{apply } \Phi} \Phi(g^{-1}) = \Phi(g^{-1})^*$, or $H = H^*$. Also, $e^{iH} = U$: $g(g^{-1}(t)) = t \xRightarrow{\text{apply } \Phi}$ $g(g^{-1}(U)) = U \Rightarrow g(H) = U$, or $e^{iH} = U$.

Connect any invertible element to $I$.

**Thm:** The set $G(B(H))$ of invertible operators in $B(H)$ is pathwise connected.

Proof: Let $T \in G(B(H))$ w/ polar decomp. $T = U|T|$. $T$ invertible $\Rightarrow |T|$ invertible, so $U = T|T|^{-1}$ is invertible. $U$ is also a p.i., hence $U$ is a unitary. So, $U = e^{iH}$ w/ $H = H^*$; then $U_t = e^{itH}$ ($t \in [0,1]$) continuously connects $U$ to $e^0 = I$.

$|T| \geq 0$, so via cont. func. calc., it corresponds to $f(\lambda) = \lambda$ w/ $\lambda \in \sigma(T) \subset [0, \infty)$ ($0$ not in b/c $T$ is invertible). So, $f(\lambda) = e^{\ln \lambda} \xRightarrow{CFC} |T| = e^{\ln|T|}$.

If $S_t = e^{t\ln|T|}$, $t \mapsto S_t$ is cont., and $S_0 = I$, $S_1 = |T|$.

So, if $\alpha(t) = e^{itH} \cdot e^{t\ln|T|}$, $\alpha : [0,1] \to G(B(H))$ cont., $\alpha(0) = I$, $\alpha(1) = U \cdot |T| = T$.

**Corollary:** If $T \in G(B(H))$, then $T = e^{S_1}e^{S_2}$ w/ $S_1, S_2 \in B(H)$. ⊕ Can use similar argument for $\mathcal{U}(B(H))$.

**Thm:** If $A$ is a $C^*$-alg, then

Note: $\mathcal{P}(C([0,1])) = \{0,1\}$, as $f^2=f=f^* \Rightarrow f:[0,1]\to\{0,1\}$

(1) Any element of $A$ is a linear comb. of 2 Hermitians of $A$

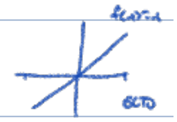
Not spanned by projections, but we can say more

(2) Any element of $A$ is a linear comb. of 4 positive elements of $A$

about the vN. alg. * This does work for $B(H)$.

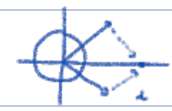
(3) Any element of $A$ is a linear comb. of 8 unitaries of $A$

Furthermore, if $A$ is a vN. alg., then $A = \overline{\mathcal{P}(A)}^{\|\cdot\|}$ (w/ $\mathcal{P}(A)$ = proj. of $A$).




**Proof:** (1) Let $T \in B(H)$. Then $T = H_1 + iH_2$ w/ $H_1 = T+T^*/2$, $H_2 = T-T^*/2i$ both Hermitian. Think $z = a+ib$ w/ $a = z+\bar{z}/2$, $b = z-\bar{z}/2i$.

(2) It suffices to show: if $T=T^*$, $T$ is a lin. comb. of 2 pos. operators in $A$. Via CFC, $T$ corresponds to $f(\lambda)=\lambda$ ($\mathrm{id}_{\sigma(T)}(\lambda)$) and $\sigma(T)\subset\mathbb{R}$. So, $f = f^+ - f^-$ w/ $f^+ = \max\{0,f\}$, $f^- = \min\{0,f\}$ both cont. and positive. So, $T = f^+(T) - f^-(T)$. Done!



(3) Let $T \geq 0$ in $A$. We write $T$ as a linear comb. of 2 unitaries in $A$. $T$ corresponds to $f(\lambda)=\lambda$ on $\sigma(T)\subset[0,\infty)$.

Further, assume $\|T\|\leq 1$ so $\sigma(T)\subset[0,1]$. Let $g_1(\lambda) = \lambda + i\sqrt{1+\lambda^2}$, $g_2(\lambda) = \lambda - i\sqrt{1+\lambda^2}$. Then $\frac{1}{2}g_1(\lambda) + \frac{1}{2}g_2(\lambda) = \lambda$, $g_1(\lambda)\overline{g_1(\lambda)} = g_2(\lambda)\overline{g_2(\lambda)} = \lambda^2 + (1-\lambda^2) = 1$.

So, $T = \frac{1}{2}g_1(T) + \frac{1}{2}g_2(T)$ w/ $g_1(T), g_2(T)$ unitaries.

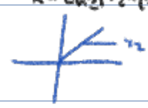
(4) Let $T\in A$, and assume $T=T^*$ (then use (1)). Borel func. calc. tells us we can identify $T$ by $f(\lambda)=\lambda$, $\lambda\in\sigma(T)$. It is known

$\Rightarrow s_n \to f$ ptwse, $s_n$ bounded

that there exist $s_n$ simple functions on $\sigma(T)$ s.t. $s_n \xrightarrow{\|\cdot\|_\infty} f$. Note $s_n$ = sum of projs in $B(\sigma(T))$ (as $\chi_E = \chi_E^2 = \overline{\chi_E}$). So, $s_n(T) \xrightarrow{\text{s.o.}} f(T) = T$

w/ $s_n(T) \in \operatorname{span}\mathcal{P}(A)$ as $\chi_E(T)$ is a proj. This shows that $A = \overline{\operatorname{span}\mathcal{P}(A)}^{\text{s.o.}}$, but why is $A = \overline{\operatorname{span}\mathcal{P}(A)}^{\|\cdot\|}$?

**Thm:** Let $A$ be a vN. alg. and $x\in A$ w/ $0\leq\|x\|\leq 1$. Then there exist $p_1, p_2, p_3, \ldots \in \mathcal{P}(A)$ with $x = \sum_{n\geq 1} \frac{1}{2^n} p_n$ (norm convergence).




**Proof:** $x$ corresponds to $\mathrm{id}(\lambda)=\lambda$ on $\sigma(x)\subset[0,1]$. Consider $\frac{1}{2}p_1 = \frac{1}{2}\chi_{[1/2,1]}(x)$. Then $\|x - \frac{1}{2}p_1\| \leq \frac{1}{2}$.

For $x - \frac{1}{2}p_1$, find $p_2$ s.t. $\|(x-\frac{1}{2}p_1) - \frac{1}{4}p_2\| \leq \frac{1}{4}$, etc. Will continue next time!