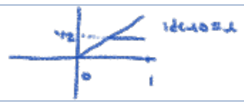
**Thm:** If $M$ is a vN alg, then $\overline{\text{span}\,\mathcal{P}(M)}^{\|\ \|} = M$. In fact, if $0 \le x \le 1$ is in $M$, then $x = \sum \frac{1}{2^n} e_n$ (norm convergence), $e_n \in \mathcal{P}(M)$.

**Remark:** In general, $C^*$-algs can be quite sparse in projections: $C([0,1])$ has only projs $0, 1$.



**Proof:** $0 \le x \le 1 \Rightarrow \sigma(x) \subset [0,1]$. Let $\frac{1}{2}e_1 = \frac{1}{2}\chi_{[1/2,1]}(x)$. Note $0 \le x - \frac{1}{2}e_1 \le \frac{1}{2}$ ($0 \le \lambda - \frac{1}{2}\chi_{[1/2,1]}(\lambda) \le \frac{1}{2}$ $\forall \lambda \in [0,1]$). Iterate: $0 \le 2x - e_1 \le 1$, so

$\exists e_2 \in \mathcal{P}(M)$ s.t. $0 \le 2x - e_1 - \frac{1}{2}e_2 \le \frac{1}{2}$ (specifically $e_2 = \chi_{[1/2,1]}(2x - e_1)$), so $0 \le x - \frac{1}{2}e_1 - \frac{1}{4}e_2 \le \frac{1}{4}$, etc. So, $0 \le x - \frac{1}{2}e_1 - \cdots - \frac{1}{2^n}e_n \le \frac{1}{2^n}$ (or $\frac{1}{2^{n-1}}$), hence

$x - \frac{1}{2}e_1 - \cdots - \frac{1}{2^n}e_n$ goes to $0$ in $\|\ \|$. Think $0.3 = \frac{1}{2}\cdot 0 + \frac{1}{4}\cdot 1 + \frac{1}{8}\cdot 0 + \cdots$

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**Thm:** If $x \in B(H)$, $x = x^*$, then $\chi_{\{0\}}(x) = \text{proj}_{\ker x}$. More generally, for $t \in \mathbb{R}$ ($t \in \sigma(x)$), then $\chi_{\{t\}}(x) = \text{proj}_{V_t}$ w/ $\ker(t \cdot I - x) =$ eigenspace of $t$.

**Proof:** Let $p = \chi_{\{0\}}(x) \in \text{v.N.}(x)$ (this is a proj.). $\text{Range}(p) = \{\xi \in H : p\xi = \xi\}$. We need to show that this is $\ker(x)$.

(i) Let $\xi \in H$ s.t. $p\xi = \xi$. We show $x\xi = 0$. $p\xi = \xi \Rightarrow xp\xi = x\xi$. $xp$ corresponds via func. calc. to $\lambda \cdot \chi_{\{0\}}(\lambda)$, which is always $0$.

Thus, $xp = 0 \Rightarrow x\xi = 0 \Rightarrow \xi \in \ker x$.

(ii) Assume $x\xi = 0$. We show $p\xi = \xi$, so $(1-p)\xi = 0$ ($\chi_{\mathbb{R}\setminus\{0\}}(x)$). $x\xi = 0 \Rightarrow x^n\xi = 0$ $\forall n \in \mathbb{N} \Rightarrow p(x)\xi = 0$ $\forall p$ polys. w/ $p(0) = 0$ $\Rightarrow$ (pointwise limit) $f(x)\xi = 0$ $\forall f$ Borel

bounded func. w/ $f(0) = 0$. So, if $f = \chi_{\mathbb{R}\setminus\{0\}}(x)$, $f(0) = 0$, so done!

**Exercise:** Any cont. func. $f$ on a compact set can be approx. by polys w/ $p(0) = 0$, if $f(0) = 0$.

Then, any bounded Borel func. $f$ w/ $f(0) = 0$ can be approx. (pointwise, by bounded seq.) by cont. func. $g$ w/ $g(0) = 0$.

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**Thm:** (Bounded Borel Functional Calculus): Let $x \in B(H)$ be normal, and denote $\mathcal{B}(\sigma(x)) = \{f : \sigma(x) \to \mathbb{C} : f \text{ bounded Borel}\}$. Then

there exists a unique $*$-morphism $\Phi : \mathcal{B}(\sigma(x)) \to \text{v.N}(x) \subset B(H)$, $f \mapsto \Phi(f) =: f(x)$ s.t.

(1) $\Phi|_{C(\sigma(x))}$ is the cont. func. calc.

(2) if $f_n \to f$ ptwise with $\sup_n \|f_n\|_\infty < \infty$, $f_n, f \in \mathcal{B}(\sigma(x))$, then $f_n(x) \xrightarrow{\text{s.o.}} f(x)$.

Moreover, $\|\Phi\| \le 1$ (so if $f_n \xrightarrow{\|\ \|_\infty} f$, then $f_n(x) \xrightarrow{\|\ \|} f(x)$, so it is cont.

In general, $\Phi$ is not 1-1, and if $H$ is separable, $\Phi$ is onto, but not otherwise.

<u>Proof:</u> We know there is an isometric $*$-morphism $\varphi$ from $C(\sigma(x))$ to $C^*(1,x)$. By (1), we want $\Phi$ to extend $\varphi$ to $\mathcal{B}(\sigma(x))$.

<u>Uniqueness of $\Phi$:</u> By a thm of Baire, any $f \in \mathcal{B}(\sigma(x))$ is the pointwise limit of a uniformly bounded seq. $f_n \in C(\sigma(x))$.

So, by (2), $f(x) = \lim^{s.o.} f_n(x) = \lim^{s.o.} \Phi(f_n) = \lim^{s.o.} \varphi(f_n)$ (if $f_n$ cont.). So, $\Phi$ is unique.

<u>Existence of $\Phi$:</u> $\Phi(f) = f(x)$ is already defined for $f \in C(\sigma(x))$.

$$C(\sigma(x)) \ni f \longmapsto \overset{\in B(H)}{f(x)} \longmapsto \langle f(x)\xi, \eta \rangle \in \mathbb{C} \quad \text{for any } \xi, \eta \in H.$$

dual ← Review

This is a cont. linear map from $C(\sigma(x))$ to $\mathbb{C}$, so in $C(\sigma(x))^*$.

<u>Thm:</u> (Riesz-Markov-Kakutani Representation Thm): If $K \subset \mathbb{C}$ is compact, any $s \in C(K)^*$ is of the form $s(f) = \int_K f \, d\mu$ for some $\mu$ complex-valued, Borel, regular measure on $K$, and $|\mu| \leq \|s\|$.

- Regular means $\mu(E) = \inf_{\substack{E \subset O \\ \text{open}}} \mu(O)$ $\forall E$ Borel.
- Complex-valued means $\mu = \mu_R + i\mu_I$ w/ $\mu_R, \mu_I$ real-valued ($\mu_R = \mu_R^+ - \mu_R^-$, pos. measures).
- Total mass/variation $|\mu| = \mu_R^+(K) + \mu_R^-(K) + \mu_I^+(K) + \mu_I^-(K)$.