Thm: Let $x \in B(H)$ normal and $\mathfrak{B}(\sigma(x)) = \{f : \sigma(x) \to \mathbb{C} \text{ bounded, Borel}\}$. Then there is a unique $*$-morphism Φ, $\|\Phi\| \le 1$, from $\mathfrak{B}(\sigma(x))$ to $v.N.(x)$ such that:

(i) $\Phi|_{C(\sigma(x))}$ is the cont. func. calculus, so $\Phi(f) = f(x) \in C^*(1, x)$

(ii) if $f_n, f \in \mathfrak{B}(\sigma(x))$, $f_n \to f$ pointwise, and $\sup_n \|f_n\|_\infty < \infty$, then $\Phi(f_n) \xrightarrow{s.o.} \Phi(f)$.

Notation: $\Phi(f)$ is denoted $f(x)$.

Remark: If $f_n \xrightarrow{\|\cdot\|_\infty} f$ then $f_n(x) \xrightarrow{\|\cdot\|} f(x)$, as $\underbrace{\|\Phi(f_n - f)\|}_{\|f_n(x) - f_n(x)\|} \le \|\Phi\| \, \|f_n - f\|_\infty \le \|f_n - f\|_\infty$.

Proof: Uniqueness was established last time.

For $\xi, \eta \in H$, we can look at $C(\sigma(x)) \ni f \xrightarrow{\alpha_{\xi,\eta}} \langle f(x)\xi, \eta \rangle \in \mathbb{C}$. $\alpha_{\xi,\eta} \in C(\sigma(x))^*$ (note $|\alpha_{\xi,\eta}(f)| \le \|f\|_\infty \|\xi\| \|\eta\|$, so $\|\alpha_{\xi,\eta}\| \le \|\xi\| \|\eta\|$).

"dual of cont. func. is measures"

By Riesz-Markov-Kakutani Representation Theorem, there is $\mu_{\xi,\eta}$ Borel regular measure on $\sigma(x)$ with $|\mu_{\xi,\eta}| \le \|\xi\| \cdot \|\eta\|$ and

$\underbrace{\alpha_{\xi,\eta}(f)}_{\langle f(x)\xi, \eta \rangle} = \int_{\sigma(x)} f \, d\mu_{\xi,\eta}$ $\forall f \in C(\sigma(x))$. So, $\forall f \in C(\sigma(x))$, $\langle f(x)\xi, \eta \rangle = \int_{\sigma(x)} f \, d\mu_{\xi,\eta}$.

claim is that this is a sesquilinear form, essentially

Claim: $(\xi, \eta) \longmapsto \int f \, d\mu_{\xi,\eta}$ (for a fixed $f \in \mathfrak{B}(\sigma(x))$) is linear in the 1st variable, conjugate linear in the 2nd variable, and it is bounded w/ norm $\le \|f\|_\infty$, hence it is of the form $(\xi, \eta) \longmapsto \langle T\xi, \eta \rangle$ for some $T \in B(H)$ w/ $\|T\| \le \|f\|_\infty$.

Review

Hence, we can define $f(x)$ (for $f \in \mathfrak{B}(\sigma(x))$) to be this unique T. So, $\langle f(x)\xi, \eta \rangle = \int f \, d\mu_{\xi,\eta}$ $\forall f \in \mathfrak{B}(\sigma(x))$.

Why is $\mu_{\xi,\eta}$ linear in ξ and conjugate linear in η? Let's check just $\mu_{\xi_1+\xi_2,\eta} = \mu_{\xi_1,\eta} + \mu_{\xi_2,\eta}$.

$\Leftrightarrow \int_{\sigma(x)} f \, d\mu_{\xi_1+\xi_2,\eta} = \int_{\sigma(x)} f \, d\mu_{\xi_1,\eta} + \int_{\sigma(x)} f \, d\mu_{\xi_2,\eta}$ $\forall f$ cont. $\Leftrightarrow \langle f(x)(\xi_1 + \xi_2), \eta \rangle = \langle f(x)\xi_1, \eta \rangle + \langle f(x)\xi_2, \eta \rangle$, which is true.

So, $\int f \, d\mu_{\xi_1+\xi_2,\eta} = \int f \, d(\mu_{\xi_1,\eta} + \mu_{\xi_2,\eta}) = \int f \, d\mu_{\xi_1,\eta} + \int f \, d\mu_{\xi_2,\eta}$ $\forall f \in \mathfrak{B}(\sigma(x))$.

So, at this stage we defined $\Phi : \mathfrak{B}(\sigma(x)) \longrightarrow v.N.(x)$, $f \longmapsto f(x)$, where $\langle f(x)\xi, \eta \rangle = \int_{\sigma(x)} f \, d\mu_{\xi,\eta}$ $\forall \xi, \eta \in H$.

<u>It remains to show:</u> (i) Φ is a $*$-morphism, $\|\Phi\| \le 1$ and (ii) $f_n \to f$ ptwise, $\sup \|f_n\|_\infty < \infty \Rightarrow \Phi(f_n) \xrightarrow{s.o.} \Phi(f)$.

Let's do (ii), assuming (i) is true: $\|f_n(x)\xi - f(x)\xi\|^2 = \langle (f_n-f)(x)\xi, (f_n-f)(x)\xi\rangle = \langle \overline{(f_n-f)}(x)\cdot(f_n-f)(x)\xi, \xi\rangle = \langle |f_n-f|^2(x)\xi, \xi\rangle$

$= \int |f_n-f|^2 d\mu_{\xi,\eta} \to 0$ by Lebesgue DCT, as $\||f_n-f|^2\|_\infty$ are all dominated by some constant k and $\int k\, d\mu_{\xi,\eta} = k\cdot\mu_{\xi,\eta}(\sigma(x)) < \infty$.

Remark: In particular, it follows that $\Phi(f) = f(x) \in v.N.(x)$ $\forall f \in \mathcal{B}(\sigma(x))$. Indeed, $\exists f_n \to f$, $f_n \in C(\sigma(x))$, pointwise and $\sup\|f_n\|_\infty < \infty$

(by a thm of Baire). So, $\overbrace{f_n(x)}^{\in C^*(1,x)} \xrightarrow{s.o.} f(x)$, hence $f(x) \in \overline{C^*(1,x)}^{s.o.} = v.N.(x)$.

Now, let's prove (i): Φ is a $*$-morphism, $\|\Phi\| \leq 1$.

• $\Phi(f+g) = \Phi(f) + \Phi(g)$? We already know this is true for $f, g \in C(\sigma(x))$. So, for $f, g \in \mathcal{B}(\sigma(x))$,

$\langle \Phi(f+g)\xi, \eta\rangle = \langle \Phi(f)\xi,\eta\rangle + \langle \Phi(g)\xi,\eta\rangle \iff \int (f+g)\, d\mu_{\xi,\eta} = \int f\, d\mu_{\xi,\eta} + \int g\, d\mu_{\xi,\eta} \Rightarrow \Phi(f+g) = \Phi(f)+\Phi(g)$. $\Phi(c\cdot f) = c\Phi(f)$ $\forall c \in \mathbb{C}$ is similar.

$\langle \xi, f(x)\eta\rangle = \overline{\langle f(x)\eta, \xi\rangle}$

• $\Phi(\bar f) = \Phi(f)^*$? (or $\bar f(x) = f(x)^*$). If $f \in C(\sigma(x))$, we know this is true: $\langle \bar f(x)\xi, \eta\rangle = \langle f(x)^*\xi, \eta\rangle$

$\iff \int \bar f\, d\mu_{\xi,\eta} = \overline{\int f\, d\mu_{\eta,\xi}} = \int \bar f\, d\bar\mu_{\eta,\xi}$ $\forall f$ cont. $\Rightarrow \mu_{\xi,\eta} = \bar\mu_{\eta,\xi} \Rightarrow \int \bar f\, d\mu_{\xi,\eta} = \int \bar f\, d\overline{\mu_{\eta,\xi}}$ $\forall f$ Borel $\Rightarrow \bar f(x) = f(x)^*$.

• $\Phi(fg) = \Phi(f)\cdot\Phi(g)$? ($(fg)(x) = f(x)g(x)$). For f, g cont. on $\sigma(x)$, $\overbrace{\langle (fg)(x)\xi, \eta\rangle}^{\int fg\, d\mu_{\xi,\eta}} = \langle f(x)g(x)\xi,\eta\rangle = \begin{cases} \langle f(x)(g(x)\xi), \eta\rangle = \int f\, d\mu_{g(x)\xi,\eta} \\ \langle g(x)\xi, f(x)^*\eta\rangle = \int g\, d\mu_{\xi, f(x)^*\eta} \end{cases}$.

So, $g\, d\mu_{\xi,\eta} = d\mu_{g(x)\xi,\eta}$ $\forall g$ cont. and $f\, d\mu_{\xi,\eta} = d\mu_{\xi, f(x)^*\eta}$ $\forall f$ cont.

If we argue that $g\, d\mu_{\xi,\eta} = d\mu_{g(x)\xi,\eta}$ works for all g Borel, we are done! (trace back steps w/ f Borel integrated w/ respect to these equal measures). We show that $\forall f$ cont., $\int fg\, d\mu_{\xi,\eta} = \int f\, d\mu_{g(x)\xi,\eta}$.

$\underbrace{\langle fg(x)\xi,\eta\rangle}_{=\int fg\, d\mu_{\xi,\eta}} \overset{?}{=} \underbrace{\langle f(x)g(x)\xi,\eta\rangle}_{=\int f\, d\mu_{g(x)\xi,\eta}}$ will finish next time!