<u>For homework:</u> $T_i \xrightarrow{\text{s.o.}} T \not\Rightarrow T_i^* \xrightarrow{\text{s.o.}} T^*$, but this is true for <u>normal</u> operators!

<u>Goal:</u> If $\|(T-S)\xi\|$ is small, then $\|(T^*-S^*)\xi\|$ is small (for T,S normal). $\|(T^*-S^*)\xi\|^2 = \cdots$ (express in terms of $(T-S)$).

Something like $TT^*-SS^* = (T-S)S^* + S(T^*-S^*)$, for example.

<u>Last time:</u> We were showing that $\tau(\mathcal{P}(\overline{\otimes} M_2(\mathbb{C})^{\text{"}})) = \{k/2^n : n \geq 0, 0 \leq k \leq 2^n\}$.

(1) If $p \in \mathcal{P}(\overline{\otimes} M_2(\mathbb{C})^{\text{"}})$, there is $n \in \mathbb{N}$ and $q \in \mathcal{P}(M_{2^n}(\mathbb{C}))$ s.t. $\|p-q\|<1$.

(2) $\|p-q\|<1 \Rightarrow q = UpU^*$ (U unitary) $\Rightarrow \tau(q) = \tau(p)$, so $\tau(p) = \tau(q) = k/2^n$.

<u>For (2):</u> If $x = qp + (1-q)(1-p)$, then $qx = xp$ and $x^*x = xx^* = 1-(p-q)^2 \Rightarrow x$ is normal and $\|1-x^*x\|<1 \Rightarrow x^*x$ is invertible (hence x is invertible via CFC) $\Rightarrow |x| = (x^*x)^{1/2}$ is invertible. So, if we do polar decomp. of x: $x = U|x|$, then $U = x|x|^{-1}$ is invertible and p.i., hence unitary. $qx = xp \overset{*}{\Rightarrow} x^*q = px^* \Rightarrow x^*xp = x^*qx = px^*x \Rightarrow x^*x \in p' \Rightarrow |x| \in p'$ (as p' is a C^*-alg and $|x| \in C^*(1, x^*x)$). So, $Up = x|x|^{-1}p = xp|x|^{-1} = qx|x|^{-1} = qU$. Hence, $q = UpU^*$.

<u>Note:</u> $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ have distance 1 and are not unitarily conjugate (so 1 is the best possible bound).

<u>Proof of (1):</u> Let $p \in \mathcal{P}(\overline{\cup M_{2^n}(\mathbb{C})}^{\text{"}})$ and $0<\varepsilon<1$. $\exists n \in \mathbb{N}, x \in M_{2^n}(\mathbb{C})$ s.t. $\underset{\substack{\Downarrow \\ \text{should be almost self-adjoint} \\ \text{\& idempotent}}}{\|p-x\|<\varepsilon}$. Let $y = {}^{x+x^*}/_2$. Then $y = y^*$, $y \in M_{2^n}(\mathbb{C})$, and $\|p-y\| = \|p-({}^{x+x^*}/_2)\| = {}^1/_2\|2p-x-x^*\| \leq {}^1/_2(\|p-x\| + \|p-x^*\|) < \varepsilon$ [$\|(p-x)^*\| = \|p-x\|$]. Note: $\|y^2-y\| \leq \|y^2-p^2\| + \|p-y\| = \|y^2-yp+yp-p^2\| + \|p-y\| \leq \|y(y-p) + (y-p)p\| + \|p-y\| \leq \varepsilon\|y\| + \varepsilon + \varepsilon = (2+\|y\|)\varepsilon \leq (2+\|x\|)\varepsilon$. Know $\|x-p\|<\varepsilon \Rightarrow \|x\| < \|p\| + \varepsilon = 1+\varepsilon$, so $(2+\|x\|)\varepsilon \leq (3+\varepsilon)\varepsilon \leq 4\varepsilon$. Now, $y = y^* \Rightarrow \exists U \in M_{2^n}(\mathbb{C})$ unitary s.t. $y = UdU^*$ with d diagonal matrix, $d = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$, and $\|y^2-y\| < 4\varepsilon \Rightarrow \|d^2-d\| < 4\varepsilon$ (since $\|UaU^*\| = \|a\|$) $\Leftrightarrow |\lambda_i^2 - \lambda_i| < 4\varepsilon$ $\forall i = 1, \ldots, n$.

<u>Note:</u> $|\lambda^2-\lambda| < 4\varepsilon \Rightarrow -4\varepsilon < \lambda^2-\lambda < 4\varepsilon \Leftrightarrow -4\varepsilon + {}^1/_4 < \lambda^2 - \lambda + {}^1/_4 < 4\varepsilon + {}^1/_4 \Leftrightarrow {}^1/_4 - 4\varepsilon < (\lambda - {}^1/_2)^2 < {}^1/_4 + 4\varepsilon \Rightarrow (\lambda-{}^1/_2)^2 < {}^1/_4 + 4\varepsilon \Rightarrow |\lambda - {}^1/_2| < \sqrt{{}^1/_4 + 4\varepsilon} < \sqrt{{}^1/_4} + \sqrt{4\varepsilon} < {}^1/_2 + 2\sqrt{\varepsilon}$.

Also, ${}^1/_4 - 4\varepsilon < (\lambda - {}^1/_2)^2$ [$\varepsilon < {}^1/_{16}$] $\Rightarrow \sqrt{{}^1/_4 - 4\varepsilon} < |\lambda - {}^1/_2|$ [$\geq \sqrt{{}^1/_4} - \sqrt{4\varepsilon}$] $\Rightarrow {}^1/_2 - 2\sqrt{\varepsilon} < |\lambda - {}^1/_2| < {}^1/_2 + 2\sqrt{\varepsilon}$ [$0 \quad {}^1/_2 \quad 1$] $\Rightarrow |\lambda| < 4\sqrt{\varepsilon}$ or $|1-\lambda| < 4\sqrt{\varepsilon}$. So, $\exists d'$ made with 1's and 0's on diag. (so projection) with $\|d-d'\| < 4\sqrt{\varepsilon}$. Hence, $\|y - \underset{\substack{\uparrow \\ \text{proj.}}}{Ud'U^*}\| = \|UdU^* - Ud'U^*\| < 4\sqrt{\varepsilon}$. $\|x - Ud'U^*\| < \varepsilon + 4\sqrt{\varepsilon} < 1$ for ε small. Done!

<u>Def:</u> A <u>II_1 factor</u> is a v.N. alg. M which is infinite dimensional and has a trace (s.o. cont., positive, faithful) and $Z(M) = \mathbb{C} \cdot I$ (a factor).

- Factors
  - Type I
    - I_{fin} — $M_n(\mathbb{C})$
    - I_∞ — $B(H)$, H inf. dim.
  - Type II
    - II_1
    - II_∞ — $II_1 \otimes B(H)$
  - Type III (no trace, not even admitting ∞ as values)

Def: R = the hyperfinite II_1 factor $= \overline{\bigotimes M_2(\mathbb{C})}^{s.o.}$.

This is isomorphic to any $\overline{M_{k_1}(\mathbb{C}) \otimes M_{k_2}(\mathbb{C}) \otimes M_{k_3}(\mathbb{C}) \otimes \cdots}^{s.o.}$ ($k_i \geq 2$) (not obvious!).

Later!

Thm: Any II_1 factor contains R.

Thm: R is a factor.

$A, \varphi \xrightarrow{GNS} \pi : A \to B(H)$, $H = \overline{A}^{\|\|_{2,\varphi}}$, $\|y\|_2 = \tau(y^*y)^{1/2}$.

Proof: Let $x \in Z(R) = R' \cap R$. We show x is a scalar (in $\mathbb{C} \cdot 1$). So, $x \in R = \overline{\bigcup M_{2^n}(\mathbb{C})}^{s.o.} \Rightarrow \exists x_i \in \bigcup M_{2^n}(\mathbb{C})$ with $x_i \xrightarrow{s.o.} x$. In particular,

$x_i \hat{1} \xrightarrow{\|\|_{2,\varphi}} x\hat{1} \Rightarrow \hat{x}_i \xrightarrow{\|\|_2} \hat{x} \Rightarrow \|x_i - x\|_2 \to 0$.

Note: $\|uy\|_2 = \|yu^*\|_2 = \|y\|_2$

Let $\varepsilon > 0$. $\exists n \in \mathbb{N}$, $a \in M_{2^n}(\mathbb{C})$ s.t. $\|a - x\|_2 < \varepsilon$. $ux = xu$ $\forall u \in \underset{\text{unitaries}}{\mathcal{U}}(M_{2^n}(\mathbb{C})) \Rightarrow uxu^* = x \Rightarrow \|uau^* - a\|_2 \leq \|uau^* - uxu^*\|_2 + \|uxu^* - a\|_2 \leq \|a - x\|_2 + \|x - a\|_2$

$< 2\varepsilon$.

Will continue next time!