$R = \overline{\cup M_{2^n}(\mathbb{C})}^{s.o.}$ has $Z(R) = \mathbb{C}\cdot I$. $\|z\|_2 = \tau(z^*z)^{1/2}$, $A, \varphi \to \pi: A \to B(H)$, $H = \overline{A}^{\|\|_{2,\varphi}}$. $A \subset \overline{A}^{s.o.} \subset \overline{A}^{\|\|_2} = H$.

$x_i \xrightarrow{s.o.} x$
$\Rightarrow x_i \hat{1} \to x \hat{1}$ in $\|\|_2$
$\Rightarrow \hat{x_i} \to \hat{x}$

Note: $\overline{A}^{s.o.} = \overline{A}^{\|\|_2}$ on bounded sets

Similar to $L(G) = \overline{\mathbb{C}[G]}^{s.o.} \subset \overline{\mathbb{C}[G]}^{\|\|_2} = \ell^2(G)$.

Let $x \in Z(R)$, and let $\varepsilon > 0$. Since $x \in R$, $\exists\, y \in M_{2^n}(\mathbb{C})$ s.t. $\|y - x\|_2 < \varepsilon$.

since $x \in Z(R)$

"y almost commutes with all unitaries"

$\forall u \in \mathcal{U}(M_{2^n}(\mathbb{C}))$, $uxu^* = x$. So, $\|uyu^* - \underbrace{uxu^*}_{=x}\|_2 = \|y - x\|_2 < \varepsilon$. $\Rightarrow \|uyu^* - x\|_2 < \varepsilon \Rightarrow \|y - uyu^*\|_2 < 2\varepsilon$.

So, if y almost commutes with all unitaries, then y is almost a scalar.

<u>Lemma:</u> If $\|y - uyu^*\|_2 < 2\varepsilon$ $\forall u \in \mathcal{U}(M_{2^n}(\mathbb{C}))$ (and $y \in M_{2^n}(\mathbb{C})$), then there is $\lambda \in \mathbb{C}\cdot I$ s.t. $\|y - \lambda\cdot I\|_2 < 2\varepsilon$.

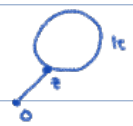
<u>Proof 1:</u> Let $G = \mathcal{U}(M_{2^n}(\mathbb{C}))$ (compact topological group), hence it has a Haar measure: $\exists \mu$ measure on G (on Borel sets) with

$\mu(G) = 1$ and μ is left-invariant: $\mu(g\cdot S) = \mu(S)$ $\forall g \in G$, $S \subset G$ measurable.

Let $f(u) = uyu^*$, $f: G \to M_{2^n}(\mathbb{C})$. Let $\lambda = \int_G f(u)\,d\mu(u) = \int_G uyu^*\,d\mu(u)$. Note: $\forall v \in G$, $v\lambda v^* = \lambda$, hence $\lambda \in \mathbb{C}\cdot I$. Indeed, $v\lambda v^* =$

$v \int uyu^*\,d\mu(u)\, v^* = \int vuyu^*v^*\,d\mu(u) = \int (vu)y(vu)^*\,d\mu(u) \overset{\text{left-invariant}}{=} \int uyu^*\,d\mu(u) = \lambda$. $u \mapsto vu$

Now, $\underbrace{\|\int_G (y - uyu^*)\,d\mu\|_2}_{=\|y-\lambda\|_2} \le \int \|y - uyu^*\|_2\,d\mu \le \int 2\varepsilon\,d\mu = 2\varepsilon\mu(G) = 2\varepsilon$. $\Rightarrow \|y - \lambda\|_2 < 2\varepsilon$.



<u>Proof 2:</u> Let K = closed convex hull of $\{uyu^* : u \in \mathcal{U}(M_{2^n}(\mathbb{C}))\} = \overline{\{c_1u_1yu_1^* + c_2u_2yu_2^* + \cdots + c_nu_nyu_n^* : 0 \le c_i \le 1,\ c_1 + \cdots + c_n = 1\}}^{\|\|_2}$.

So, there is a unique element $z \in K$ of minimal $\|\|_2$. In particular, $\|uzu^*\|_2 = \|z\|_2$ and $uzu^* \in K$ (as $uKu^* = K$), hence $uzu^* = z$

$\forall u \in \mathcal{U}(M_{2^n}(\mathbb{C}))$ (by uniqueness). So, z commutes with all $\mathcal{U}(M_{2^n}(\mathbb{C}))$, hence z is a scalar.

Note that $\|y - z\|_2 < 2\varepsilon$, because for any element $T \in K$, we have $\|y - T\|_2 < 2\varepsilon$: $T = c_1\alpha_1 + \cdots + c_n\alpha_n$, $\alpha_i = u_iyu_i^*$, so $\|y - \alpha_i\|_2 < 2\varepsilon$

$\Rightarrow \|y - T\|_2 \le \|(c_1 + \cdots + c_n)y - c_1\alpha_1 - \cdots - c_n\alpha_n\|_2 \le c_1\|y - \alpha_1\|_2 + \cdots + c_n\|y - \alpha_n\|_2 < (c_1 + \cdots + c_n)2\varepsilon = 2\varepsilon$.

<u>v.N. Examples</u>

(I) Finite dimensional: $M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$.

(II) Abelian: $L^\infty(X, \mu)$

(III) Group-based: $L(G)$ (More generally: group-measure space $G \ltimes L^\infty(X, \mu)$)

(IV) $R = \overline{\cup M_{2^n}(\mathbb{C})}^{s.o.}$ (Also: GNS-based examples)

Borel Functional Calculus

Notation: If $Y \subset \mathbb{C}$ is a compact set, then $\mathcal{B}(Y)$ = all bounded Borel measurable functions on Y.

$C(Y) \subset \mathcal{B}(Y)$ (also contains χ_E's!)

Thm: Let $x \in B(H)$ be a normal operator. There exists a unique unital $*$-homomorphism $\Phi : \mathcal{B}(\sigma(x)) \to vN(x) \subset B(H)$ with $\|\Phi\| \leq 1$

such that:

(1) Φ extends the cont. functional calculus, i.e. $\Phi|_{C(\sigma(x))} : C(\sigma(x)) \to C^*(1,x)$ is the cont. functional calculus

$*$ isomorphism

$\|\ \|_\infty \qquad \downarrow \qquad \|\ \|$

(2) if $f_n \xrightarrow{\text{pointwise}} f$ are all in $\mathcal{B}(\sigma(x))$ and are uniformly bounded, then $\Phi(f_n) \xrightarrow{s.o.} \Phi(f)$.

$\exists M > 0,\ \|f_n\|_\infty \leq M$

Notation: $\Phi(f) = f(x)$.

What does Φ do?

$\lambda = id \xmapsto{\Phi} \Phi(id) = id(x)$, so x $\qquad \chi_E(\lambda) \longmapsto \chi_E(x)$ projection

$\lambda^2 \longmapsto x^2$

$\bar{\lambda} \longmapsto x^*$

$\lambda^2 \bar{\lambda} \longmapsto x^2 x^*$

$\uparrow$

all CFC

1st homework:

$0 \leq q \leq 1 \Rightarrow 0 \leq pqp \leq p1p \leq p \leq 1 \Rightarrow (pqp)^n \to$ proj.

d

Example: If $0 \leq x \leq 1$, $x \in B(H)$, show x^n converges s.o. to a projection.

Solution: Via Borel Func. Calc., we must show that $id(x)^n$ (denote it by $\lambda \mapsto \lambda$) converges pointwise to a char. func on $\sigma(x)$.

$\lambda \in [0,1]$, so $\lambda^n \to \begin{cases} 0 & \text{if } \lambda < 1 \\ 1 & \text{if } \lambda = 1 \end{cases}$. So, $\lambda^n \to \chi_{\{1\}}(\lambda)$. Done!

So, $x^n \xrightarrow{s.o.} p$ and $\lambda^n \to \chi_{\{1\}}(\lambda)$. Note $\lambda \cdot \chi_{\{1\}}(\lambda) = \chi_{\{1\}}(\lambda)$, so $xp = p$.