End of proof of Borel func. calc:

Recall: For x normal, $f \in \mathcal{B}(\sigma(x))$, $\langle f(x)\xi, \eta\rangle = \int_{\sigma(x)} f \, d\mu_{\xi,\eta}$. We have to show $f \mapsto f(x)$ is multiplicative: $(fg)(x) = f(x)g(x)$.

For f, g cont., we know it by cont. func. calc, so $\langle fg(x)\xi, \eta\rangle = \langle f(x)g(x)\xi, \eta\rangle$ $\forall \xi, \eta \in H$.

$$\Rightarrow \quad \begin{array}{c} \int fg \, d\mu_{\xi,\eta} = \langle fg(x)\xi, \eta\rangle = \\ \shortparallel \\ \int gf \, d\mu_{\xi,\eta} \end{array} \left\{ \begin{array}{l} \langle f(x)g(x)\xi, \eta\rangle = \int f \, d\mu_{g(x)\xi,\eta} \quad \forall f, g \in C(\sigma(x)) \\ \\ \langle g(x)\xi, \overline{f}(x)\eta\rangle = \int g \, d\mu_{\xi, \overline{f}(x)\eta} \end{array} \right. \Rightarrow \left\{ \begin{array}{ll} g \, d\mu_{\xi,\eta} = d\mu_{g(x)\xi,\eta} & (1) \\ \\ f \, d\mu_{\xi,\eta} = d\mu_{\xi, \overline{f}(x)\eta} & (2) \end{array} \right.$$

So far we know these for f, g cont. on $\sigma(x)$. If we can extend (1) to all g Borel, we will be done.

To show $g \, d\mu_{\xi,\eta} = d\mu_{g(x)\xi,\eta}$ (g Borel) is equiv. to $\int fg \, d\mu_{\xi,\eta} = \int f \, d\mu_{g(x)\xi,\eta}$ $\forall f \in C(\sigma(x))$ $\Leftrightarrow$ $\langle (fg)(x)\xi, \eta\rangle = \langle f(x)g(x)\xi, \eta\rangle$.

Indeed, $\langle f(x)g(x)\xi, \eta\rangle = \langle g(x)\xi, \overline{f}(x)\eta\rangle = \int g \, d\mu_{\xi, \overline{f}(x)\eta} \overset{(2)}{\underset{f \text{ cont.}}{=}} \int gf \, d\mu_{\xi,\eta} = \int fg \, d\mu_{\xi,\eta} = \langle (fg)(x)\xi, \eta\rangle$.

So, we showed $g \, d\mu_{\xi,\eta} = d\mu_{g(x)\xi,\eta}$ $\forall g \in \mathcal{B}(\sigma(x))$. Hence, for f Borel, it follows $\int fg \, d\mu_{\xi,\eta} = \int f \, d\mu_{g(x)\xi,\eta}$ $\Leftrightarrow$ $\langle (fg)(x)\xi, \eta\rangle = \langle f(x)g(x)\xi, \eta\rangle$

$\forall \xi, \eta \in H$. Finally, $(fg)(x) = f(x)g(x)$. Done!

von Neumann's Bicommutant Theorem

Thm: Let $M \subset B(H)$ be a unital, $*$-closed algebra. Then M is a v.N. algebra (so $\overline{M}^{s.o.} = M$) if and only if $M = M''$.

(More generally: if M is any unital $*$-alg. of $B(H)$, then $\overline{M}^{s.o.} = M''$.)

Examples: (1) $H = \mathbb{C}^2$, $B(H) \cong M_2(\mathbb{C})$, $M = \left\{ \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix} : c, d \in \mathbb{C} \right\}$. $M = \overline{M}^{s.o.}$, so we expect $M = M''$. In fact, $M = M'$ in this case, so $M'' = (M')' = M' = M$.

(2) If $M \subset B(H)$ is any MASA (like L^∞ in $B(L^2)$), then $M' = M \Rightarrow M'' = M \Rightarrow M$ is a v.N. alg.

(3) $M = \mathbb{C} \cdot I \subset B(H)$, $M' = B(H)$, $M'' = B(H)' = \mathbb{C} \cdot I$. $\Rightarrow M = M''$.

Exercise

(4) $H = \mathbb{C}^4$, $B(H) \cong M_4(\mathbb{C})$, $M = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in M_2(\mathbb{C}) \right\}$. M is fin. dim., so it is s.o. closed. Let's check that indeed $M = M''$.

Exercise

$M = M_2(\mathbb{C}) \otimes I \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) = M_4(\mathbb{C})$, and $(M_2(\mathbb{C}) \otimes I)' = (I \otimes M_2(\mathbb{C})) = \left\{ \begin{pmatrix} c & 0 & d & 0 \\ 0 & c & 0 & d \\ e & 0 & f & 0 \\ 0 & e & 0 & f \end{pmatrix} \right\} = M_2(\mathbb{C} I_2)$.

So, $M'' = (M')' = (M_2(\mathbb{C} I_2))' = \left\{ \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} : A \in M_2(\mathbb{C}) \right\} = M_2(\mathbb{C}) \otimes I$.

Such k is called a reducing subspace for x.

p would look like $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

The thm says that x:
$$\begin{tikzcd} k \arrow[r, "x|_k"] & k \\ \oplus & \oplus \\ k^\perp \arrow[r, "x|_{k^\perp}"'] & k^\perp \end{tikzcd}$$
. So, x as a matrix looks like $\begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ w/ $T_1 = x|_k$, $T_2 = x|_{k^\perp}$.

Proof of lemma:

Assume (3): $xp = px$. For $\xi \in k$, $xp\xi = px\xi \iff x\xi = p(x\xi) \in k \Rightarrow x\xi \in k \Rightarrow x(k) \subset k$. For $\xi \in k^\perp$, $xp\xi = px\xi \Rightarrow 0 = p(x\xi) \Rightarrow x\xi \perp k \Rightarrow x\xi \in k^\perp \Rightarrow x(k^\perp) \subset k^\perp$.

So, (3) $\Rightarrow$ (1). Similarly, (3) $\Rightarrow$ (2): $xp = px \Rightarrow px^* = x^*p \Rightarrow x^*(k) \subset k$ similarly.

(1) $\Rightarrow$ (3): $xk \subset k$, $xk^\perp \subset k^\perp$. Want: $xp\xi = px\xi$ $\forall \xi \in H$. Since $H = k \oplus k^\perp$, it suffices to check for $\xi \in k$ and $\xi \in k^\perp$.

For $\xi \in k$, $\overset{x\xi}{\overset{\shortparallel}{xp\xi}} = px\xi$ is true as $x\xi \in k \Rightarrow p(x\xi) = x\xi$. $\xi \in k^\perp$ is similar: $\overset{0}{\overset{\shortparallel}{xp\xi}} = \overset{0}{px\xi}$ as $x\xi \in k^\perp \Rightarrow p(x\xi) = 0$.

(2) $\Rightarrow$ (3): $xk \subset k$, $x^*k \subset k$. Want: $xp\xi = px\xi$ $\forall \xi \in H$. Check: for $\xi \in k$, $x\xi = p(x\xi)$ true (as $x\xi \in k$). For $\xi \in k^\perp$, $xp\xi = px\xi$? $\iff 0 = px\xi \iff x\xi \perp k$

$\iff \langle x\xi, \eta \rangle = 0$ $\forall \eta \in k \iff \langle \xi, x^*\eta \rangle = 0$ $\forall \eta \in k$, as $\overset{x^*k \subset k}{\overset{\Downarrow}{x^*\eta \in k}}$ and $\xi \in k^\perp$.

Bicommutant Thm: $M = M'' \iff M = \overline{M}^{s.o.}$.

"$\Rightarrow$" is clear b/c any commutant is s.o. closed. Think: $xy_i = y_i x \xrightarrow{s.o.} xy = yx$ if $y_i \xrightarrow{s.o.} y$.

"$\Leftarrow$": We start with $M = \overline{M}^{s.o.}$ and must show $M = M''$. $M \subset M''$ is always true.

To show that $x \in M'' \Rightarrow x \in M$, we show $x \in M'' \Rightarrow x \in \overline{M}^{s.o.}$, so any open nbd of x (s.o.) intersects M. ☆ Review s.o. topology!