<u>Last time:</u> For M v.N. alg. $p,q \in \mathcal{P}(M)$, we say $p \sim q$ if $\exists v \in M$ s.t. $v^*v = p$, $vv^* = q$.

We may also write $p \overset{v}{\sim} q$.

<u>Example:</u> $M = \mathbb{C} \oplus \mathbb{C} \subset B(\mathbb{C}^2) = M_2(\mathbb{C})$, $M = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x,y \in \mathbb{C} \right\}$.

$p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in M$ but $p \not\sim q$ in M as $p = vv^*$, $q = v^*v$ w/ $v \in M \Rightarrow v, v^*$ commute $\Rightarrow p = q$.

However, $p \sim q$ in $M_2(\mathbb{C})$, as $p = vv^*$, $q = v^*v$ w/ $v = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Similar to putting bijections together to make a larger bijection

<u>Prop:</u> If $(p_i)_{i \in I}, (q_i)_{i \in I} \in \mathcal{P}(M)$ w/ both (p_i) and (q_i) mutually orthogonal, and $p_i \sim q_i$ $\forall i$, then $\sum_{i \in I} p_i \sim \sum_{i \in I} q_i$.

<u>Remark:</u> $\sum p_i = \bigvee p_i = \lim^{s.o.} \sum_{\text{finite}} p_i \in M$. *Limit exists b/c projections are mutually orthogonal

$p_1 \xrightarrow{v_1} q_1$

<u>Proof:</u> Say $p_i \overset{v_i}{\sim} q_i$ (so $v_i v_i^* = q_i$, $v_i^* v_i = p_i$). then $p = \sum p_i \overset{v}{\sim} q = \sum q_i$ w/ $v = \sum v_i = \lim^{s.o.} \sum_{\text{finite}} v_i$. $p_2 \xrightarrow{v_2} q_2$

Note that $v\xi = v_i\xi$ for $\xi \in p_i(H)$ and $v\xi = 0$ for $\xi \in (\sum p_i)(H)^{\perp}$. Need to check: $vv^* = q$, $v^*v = p$. $v_j^* v_i = 0$ if $i \neq j$, as v_j^* is 0 on $v_i(H) = q_i(H)$

Let's do the second one: for $\xi \in p_i(H)$ (i fixed), $v^*v\xi = v^*v_i\xi = (\sum_j v_j^*) v_i \xi = \lim^{s.o.} (\sum_{\text{finite}} v_j^*) v_i \xi = v_i^* v_i \xi = p_i \xi = (\sum_i p_i)\xi$. So, $v^*v = \sum p_i$.

"p is equivalent to a subprojection of q." Think about set card. equiv. to a subset

<u>Def:</u> For $p,q \in \mathcal{P}(M)$, M v.N. alg., we say $p \precsim q$ if there is $q_1 \leq q$ s.t. $p \sim q_1$ (in M).

no subprojections

<u>Remark:</u> Our prev. example $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ in $M = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x,y \in M \right\}$ shows that not all projections are comparable: $p \not\precsim q$, $q \not\precsim p$.

$p \in \mathcal{P}(M)$, $p \neq 0,1$, $1-p \in \mathcal{P}(M)$ also.

<u>Heading towards:</u> Comparison Thm: If M is a factor, then $\forall p,q \in \mathcal{P}(M)$, $p \precsim q$ or $q \precsim p$.

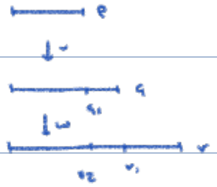
<u>Thm:</u> $\precsim$ has the properties (on $\mathcal{P}(M)$, M v.N alg). So, $\precsim$ is a partial order on eq. classes.

(i) $p \precsim p$ (iii) $p \precsim q$ and $q \precsim p \Rightarrow p \sim q$.

(ii) $p \precsim q$ and $q \precsim r \Rightarrow p \precsim r$

Proof: (i) is obvious $(v=p)$.

$q_1 \neq vp = vv^*v = v$, not q_1!

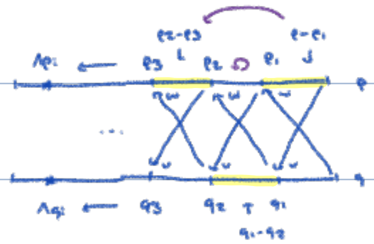


(ii) $p \precsim q \Rightarrow p \overset{v}{\sim} q_1 \leq q$ We show $\exists r_2 \leq r_1 (\leq r)$ s.t. $p \overset{wv}{\sim} r_2$. $p \overset{v}{\sim} q_1 \Rightarrow p = v^*v,\ q_1 = vv^* = vpv^*$ $(vpv^* = vv^*vv^* = q_1q_1 = q_1)$.

$q \precsim r \Rightarrow q \overset{w}{\sim} r_1 \leq r$ Define $r_2 = wq_1w^* \leq wqw^* = r_1$ and $p \overset{wv}{\sim} r_2$. Indeed, $(wv)(wv)^* = wvv^*w^* = wq_1w^* = r_2$.

$(wv)^*(wv) = v^*w^*wv = v^*qv = v^*pv = p$.

as $v^*(q-p)v = 0$ b/c $v^*|_{(q-p)(H)} = 0$



(iii) This is a v.N. version of Cantor-Schröder-Bernstein. Assume that $p \precsim q$ and $q \precsim p$.

So, $p \overset{v}{\sim} q_1 \leq q$ and $q \overset{w}{\sim} p_1 \leq p$. As $q_1 \leq q$, $q_1 \overset{w}{\sim} p_2 \leq p_1$ w/ $p_2 = wq_1w^*$.

As $p_1 \leq p$ and $p \overset{v}{\sim} q_1$, it follows $p_1 \overset{v}{\sim} q_2 \leq q_1$ w/ $q_2 = vp_1v^*$.

We show $p \sim p_1$ (hence $p \sim p_1 \overset{w^*}{\sim} q$). $p - p_1 \overset{wv}{\sim} p_2 - p_3 \overset{wv}{\sim} p_4 - p_5 \sim \cdots$ and $p = (p-p_1) + (p_1-p_2) + (p_2-p_3) + (p_3-p_4) + \cdots + \wedge_{i \geq 1} p_i$.

$$p_1 = \overbrace{(p_2-p_3)}^{wv(p-p_1)} + \overbrace{(p_1-p_2)}^{p_1-p_2} + \overbrace{(p_4-p_5)}^{wv^*(p_1-p_2)} + \overbrace{(p_3-p_4)}^{p_3-p_4} + \cdots + \overbrace{\wedge_{i \geq 2} p_i}$$

Comments on a HW problem

v.N. Ergodic Thm: If $\|T\| \leq 1$ $(T \in B(H))$, then $\frac{I+T+T^2+\cdots+T^{n-1}}{n} \xrightarrow{s.o.} p$ w/ $p = \mathrm{proj}_{\{v \in H : Tv = v\}}$.

Particular case: When $H = \mathrm{span}\{e_1,\dots,e_n\}$ and T permutes $\{e_1,\dots,e_n\}$.