Plan: Prove Comparison Thm. in factors: if $p,q\in P(M)$, then $p\precsim q$ or $q\precsim p$.

This will allow us to prove many results by playing Minecraft in a factor (breaking things into blocks).

$\ker x \xrightarrow{v} 0$

Recall: If $x\in B(H)$, there is a unique p.i. $v\in B(H)$ w/ $x=v|x|$ and $\ker v=\ker x$. $\quad (\ker x)^{\perp} \xrightarrow{v} \overline{\mathrm{Range}(x)}$ bijection, $v(|x|\xi)=x\xi$.

$(\ker x)^{\perp} = (\ker|x|)^{\perp}$

$v^*v=\mathrm{proj}_{(\ker x)^{\perp}}$, $vv^*=\mathrm{proj}_{\overline{\mathrm{Range}(x)}}$.

Thm: If M is a vN. alg. and $x\in M$, then the p.i. v from the polar decomposition of x is in M.

So, $v^*v=\mathrm{proj}_{(\ker x)^{\perp}}$, $vv^*=\mathrm{proj}_{\overline{\mathrm{Range}(x)}}\in M$ also.

Proof: $x=v|x|$, $x\in M$, and $|x|=(x^*x)^{1/2}\in C^*(1,x)\subseteq vN(x)\subseteq M$. To show that $v\in M$, we show $v\in M''=M$. So, we show v commutes w/ M'.

It suffices to show $\forall u\in M'$ unitary, $uv=vu$. Let $u\in M'$ unitary, so $uxu^*=x$ and $u|x|u^*=|x|$. $x=v|x| \Rightarrow uxu^*=(uv)|x|u^*$

$\Rightarrow (uxu^*)=(uvu^*)|x| \Rightarrow x=(uvu^*)|x|$. Note $\ker uvu^*=\ker uxu^*=\ker x$. So, uvu^* satisfies the polar dec. definition for x, hence

by uniqueness $uvu^*=v$. $\Rightarrow uv=vu$. Done!

Def: If $x\in B(H)$, then $L(x)=\mathrm{proj}_{\overline{\mathrm{Range}(x)}}$ and $R(x)=\mathrm{proj}_{(\ker x)^{\perp}}$ are called the left and right support of x.

Note: I is the biggest that works

Prop: $L(x)$ is the smallest proj. p in $B(H)$ w/ $px=x$ and $R(x)$ is the smallest proj. q in $B(H)$ w/ $xq=x$.

Proof: For p proj., $px=x \Leftrightarrow px\xi=x\xi\ \forall\xi\in H \Leftrightarrow x\xi\in p(H)\ \forall\xi\in H \Leftrightarrow \mathrm{Range}(x)\subseteq p(H) \Leftrightarrow \overline{\mathrm{Range}(x)}\subseteq p(H) \Leftrightarrow \mathrm{proj}_{\overline{\mathrm{Range}(x)}}\le p$.

$xq=x \Leftrightarrow x(1-q)=0 \Leftrightarrow x(1-q)\xi=0\ \forall\xi\in H \Leftrightarrow (1-q)\xi\in\ker x\ \forall\xi\in H \Leftrightarrow 1-q\le \mathrm{proj}_{\ker x} \Leftrightarrow q\ge 1-\mathrm{proj}_{\ker x}=\mathrm{proj}_{(\ker x)^{\perp}}$.

So, if $x\in M$ vN. $\Rightarrow L(x),R(x)\in M$. Also, $L(x)\sim R(x)$ via the p.i. from the polar dec. of x.

Compare $\underbrace{M_2\oplus M_2}_{\text{not factor}}=\left\{\begin{pmatrix} A & 0\\ 0 & B\end{pmatrix}\right\}$ to $\underbrace{M_4}_{\text{factor}}$.

Thm: (Factors have corners): If M is a factor and $p,q\in P(M)$ nonzero, then $pMq\neq 0$.

Example: $M=M_2(\mathbb{C})\subseteq B(\mathbb{C}^2)$, $p=\begin{pmatrix}1&0\\0&0\end{pmatrix}$, $q=\begin{pmatrix}0&0\\0&1\end{pmatrix}$. $pMq=\left\{\begin{pmatrix}0&x\\0&0\end{pmatrix}: x\in\mathbb{C}\right\}\neq 0$.

Remark: If M is a v.N. alg. which is not a factor, then $\exists\, p,q \in \mathcal{P}(M)$ nonzero w/ $pMq = 0$.

Indeed, $Z(M)$ is a v.N. alg., so it is spanned by its projections. So, if $Z(M) \neq \mathbb{C}\cdot I$, then $\exists\, p \in \mathcal{P}(Z(M))$ w/ $p \neq 0,1$. Let $q := 1-p \neq 0,1 \in Z(M)$.

Then $pMq = pqM = 0$ as $pq = 0$.

$$M = \begin{array}{cc} & pMq \\ \left(\begin{array}{cc} pMp & 0 \\ 0 & qMq \end{array}\right) & \\ qMp & \end{array}$$

, for $x \in M$, $x = 1\cdot x\cdot 1 = (p+q)x(p+q) = pxp + qxq + \overbrace{pxq + qxp}^{0}$.

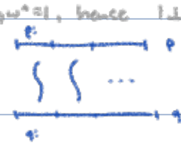
* You can break v.N. algs. into a direct sum/integral of factors.

Proof of thm: Assume, by contradiction, that $pMq = 0$, so $pxq = 0\ \forall x \in M$. Let $u,w \in U(M)$. Then $\underbrace{upu^*(wqw^*) = 0}_{pxq=0 \text{ w/ } x = u^*w} \Rightarrow upu^* \perp wqw^*$.

$\Rightarrow \bigvee_{u\in U(M)} upu^* \perp \bigvee_{w \in U(M)} wqw^*$. Let $S = \bigvee_{u \in U(M)} upu^*$. Then $u_0 S u_0^* = S\ \forall u_0 \in U(M)$, b/c $u_0\left(\bigvee_{u\in U(M)} upu^*\right)u_0^* = \bigvee_{u \in U(M)} u_0upu^*u_0^* = \bigvee_{(u_0u)\in U(M)} (u_0u)p(u_0u)^*$

$= \bigvee_u upu^* = S$. So, S commutes with $u_0\ \forall u_0 \in U(M)$. Since M is a factor, S must be 1. Similarly, $\bigvee_w wqw^* = 1$, hence $1 \perp 1$. $\Rightarrow\Leftarrow$

~ Compare to sets



Thm: (Comparison Thm. in Factors) If M is a factor and $p,q \in M$, then $p \precsim q$ or $q \precsim p$.

Proof: Let $\underbrace{\mathcal{F}}_{\text{pairs of families}} = \{((p_i)_{i\in I}, (q_i)_{i \in I}) \in \mathcal{P}(M) \text{ nonzero}\}$ w/ p_i mutually orthogonal $p_i \leq p$, q_i mutually orthogonal $q_i \leq q$, and $p_i \sim q_i\ \forall i$.

$\mathcal{F}$ has a natural partial order relation: $((p_i)_{i\in I}, (q_i)_{i \in I}) \leq ((p_j')_{j\in J}, (q_j')_{j\in J})$ if $I \subseteq J$, $(p_i)_{i\in I} \subset (p_j')_{j \in J}$, $(q_i)_{i\in I} \subset (q_j')_{j\in J}$.

$\mathcal{F} \neq \emptyset$: $pMq \neq 0$, so pick $y = pxq \neq 0$ w/ $x \in M$. Note $py = y \Rightarrow L(y) \leq p$ and $yq = y \Rightarrow R(y) \leq q$, but $\overset{p_1}{L(y)} \sim \overset{q_1}{R(y)}$. So, $\exists\, p_1 \leq p, q_1 \leq q$ s.t. $p_1 \sim q_1$ nonzero.

By Zorn's Lemma, $\mathcal{F}$ has a maximal element, call it $((p_i)_{i\in I}, (q_i)_{i\in I})$. Note b/c $p_i \sim q_i \Rightarrow \sum p_i \sim \sum q_i$, if $\sum p_i = p$, we are done: $p \precsim q$. If $\sum q_i = q$, we are also done: $q \precsim p$. If neither, then $\sum p_i =: S < p$ and $\sum q_i =: T < q$ w/ $p - S \neq 0$, $q - T \neq 0$. So, $\exists\, p_0 \leq p - S$, $q_0 \leq q - T$ w/ $p_0, q_0 \neq 0$ and $p_0 \sim q_0$.

However, then $((p_i)_{i \in I\cup\{0\}}, (q_i)_{i\in I \cup \{0\}}) \in \mathcal{F}$ contradicts maximality.