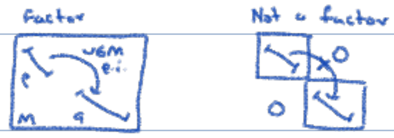
Plan: This week lectures, next week presentations.

Recall: If M is a factor (vN. alg. of trivial centre), then:



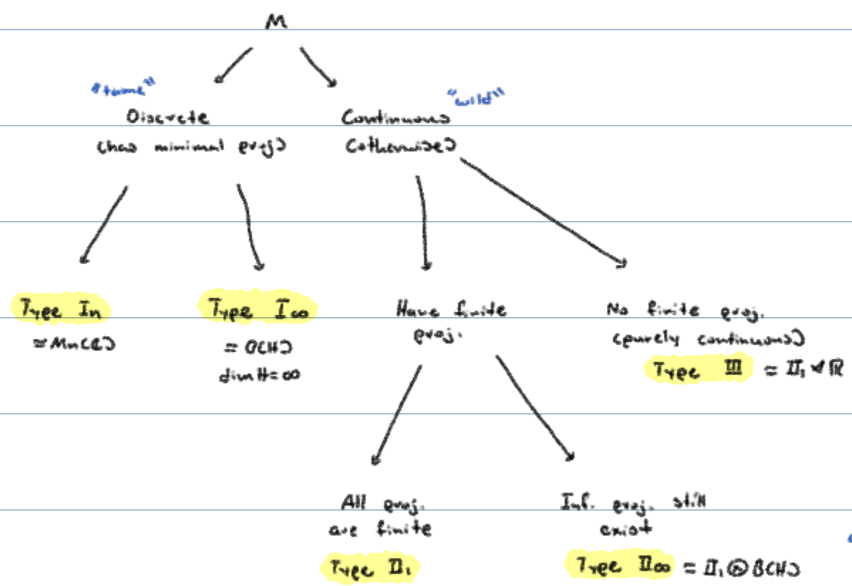
not a factor
e.g. $\chi_{[0,1/2]} L^\infty([0,1], m) \chi_{[1/2,1]} = 0$

(I) Comparison Thm: $\forall p, q \in \mathcal{P}(M)$, $p \precsim q$ or $q \precsim p$.

or no holes

(II) Factors have no nontrivial corners: $pMq \neq 0$ $\forall p, q \in \mathcal{P}(M)$ nonzero.

Classification of Factors



These can be expressed in terms of II_1 factors, so II_1 factors are the ones that really matter.

M vN. alg.

Def: $p \in \mathcal{P}(M)$ is a minimal proj. if $p \neq 0$ and $\forall q \in \mathcal{P}(M)$, $q \leq p \Rightarrow q = 0$ or $q = p$.

Examples: (1) In $L^\infty([0,1], m)$, projs. are χ_E (E measurable in $[0,1]$). If $p = \chi_E$, $m(E) \neq 0$, $\exists F \subset E$, $0 < m(F) < m(E)$, so $0 < q = \chi_F < p$. So, no minimal projections.

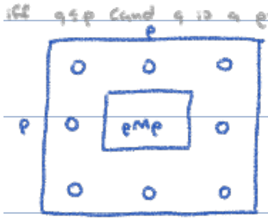
$\in \{0, 1/2, 1\}$ $\quad = 1/2$

(2) In $\mathcal{B}(\mathbb{C}^2) = M_2(\mathbb{C})$, $p = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is minimal (if $0 < q < p$, then $0 < \tau(q) < \tau(p)$; impossible!)

(3) In $R = \overline{\bigotimes}^{s.o.} M_2(\mathbb{C}) = \overline{\bigcup M_{2^n}(\mathbb{C})}^{s.o.}$ no proj. is minimal! Say $p \in \mathcal{P}(R)$, $p \neq 0 \Rightarrow \tau(p) \neq 0 \Rightarrow \exists n$ s.t. $0 < 1/2^n < \tau(p)$. $\Rightarrow \exists q \in M_{2^n}(\mathbb{C}) \subset R$ proj. s.t. $0 < \tau(q) < \tau(p)$. By Comparison Thm, $q \precsim p$ or $p \precsim q$. $q \precsim p$ means $q \sim p_1 \leq p \Rightarrow \tau(q) = \tau(p_1) < \tau(p) \Rightarrow 0 < p_1 \neq p$, so p not minimal.

Remark: $p \in \mathcal{P}(M)$ is minimal iff $pMp = \mathbb{C}p$.

Proof: pMp is a v.N. alg. (w/ unit p). Note: $M\subseteq B(H) \Rightarrow pMp \subseteq B(pH)$. $pMp=\mathbb{C}p \iff pMp$ has no proj. (except $0, p$). Note: q is a proj. in pMp iff $q\leq p$ (and q is a proj. in M). $q\leq p \iff qp=q \iff pqp=q \iff q\in pMp$.



$p=\begin{pmatrix}0&&\\&1&\\&&0\end{pmatrix}\in M_n(\mathbb{C})$

Thm: If M is a factor with at least a minimal proj. then $M\cong B(H)$.

Proof: First, let's understand the algebraic structure of $B(H)$. If we fix an ONB $(f_i)_{i\in I}$ of H, then $B(H)$ is generated by $E_{ij}:H\to H$, the p.i.'s taking $\mathbb{C}e_j$ to $\mathbb{C}e_i$ and $(\mathbb{C}e_j)^\perp$ to 0 (i.e. E_{ij} has a 1 on position (i,j) as a matrix in basis $(f_i)_{i\in I}$).

For $H=\mathbb{C}^2$, $B(\mathbb{C}^2)=M_2(\mathbb{C})=\text{span}\{\underset{E_{11}}{\begin{pmatrix}1&0\\0&0\end{pmatrix}},\underset{E_{12}}{\begin{pmatrix}0&1\\0&0\end{pmatrix}},\underset{E_{21}}{\begin{pmatrix}0&0\\1&0\end{pmatrix}},\underset{E_{22}}{\begin{pmatrix}0&0\\0&1\end{pmatrix}}\}$. Note $E_{ii}=f_i$. For $T\in B(H)$, $T\overset{\text{uniquely}}{=}\sum x_{ij}E_{ij}$ w/ $x_{ij}\in\mathbb{C}$ (the sum is a limit s.o. of partial finite sums). Moreover, $E_{kl}E_{ij}=\delta^i_l E_{kj}$. For $B(H)$, $\boxed{E_{11}}\boxed{E_{12}}\boxed{E_{13}}\cdots$

Let M be a factor which has a minimal proj. Let $\mathcal{F}=\{(p_i)_{i\in I}: p_i\in\mathcal{P}(M),\ p_i \text{ mutually orthogonal},\ p_i \text{ minimal}\}$. $\mathcal{F}$ has an obvious partial order $(\subseteq)$ and $\mathcal{F}\neq\emptyset$, so by Zorn's Lemma $\mathcal{F}$ has a maximal element $(e_i)_{i\in I}$. We claim $\sum_{i\in I}e_i=1$. If not, $f=1-\sum e_i$ is a proj. w/ $0\neq f\in M$. By Comparison Thm, $e_1\precsim f$ or $f\precsim e_1$. So, if $e_1\sim f'\leq f$, then f' is minimal. So, $\{e_i\}_{i\in I}\cup\{f'\}$ is a bigger family, $\Rightarrow\Leftarrow$.

If $f\precsim e_1$, then $f\sim f'\leq e_1$. e_1 is min, so $f'=e_1 \Rightarrow f$ min.

Fix some $i_0\in I$. For any $i\in I$, $e_{i_0}\sim e_i$ (being minimal), so $\exists v_i$ p.i. s.t. $e_i=v_iv_i^*$, $e_{i_0}=v_i^*v_i$. Will finish next time!