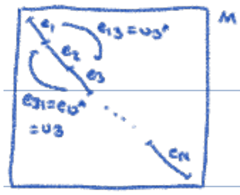
__Thm:__ If $M$ is a factor and it is discrete (i.e. has minimal projections), then $M \cong B(H)$.

__Proof:__ Let's first do the case when $M$ is finite dim. Let $(e_i)_{i\in I}$ be a maximal family of mutually orthogonal minimal projections in $M$ (exist by Zorn's Lemma). We may assume $I=\{1,\dots,N\}$ for some $N\geq 1$. Also, note that $e_1+\dots+e_N=1$. Indeed, if $f=1-(e_1+\dots+e_N)$ is a proj. $\neq 0$, then $e_1 \precsim f$ or $f \precsim e_1$. If $e_1 \precsim f$, then $f$ contains a min. proj. $\sim e_1$, call it $e_{N+1}$, which contradicts maximality of $\{e_1,\dots,e_N\}$. If $f \precsim e_1$, then $f\sim e_1$, and we can use the same argument. Thus, $e_1+\dots+e_N=1$.




$v_1 = e_1$

Since $e_1\sim e_2\sim\dots\sim e_N$ (as they are minimal), there exist p.i. $(v_i)_{i\in\{1,\dots,N\}}$ w/ $e_i \overset{v_i}{\sim} e_1$, so $e_i=v_i^*v_i$, $e_1=v_iv_i^*$.

Let $e_{1i}=v_i$ and $e_{i1}=v_i^*$, and let $e_{ij}=e_{i1}e_{1j}=v_i^*v_j$.

Compare to $M_n(\mathbb{C})$


We show that there is a $*$-morphism from $M$ to $M_n(\mathbb{C})$ taking $e_{ij}$ to $E_{ij}$, w/ $E_{ij}$ = matrix unit w/ 1 on entry $(i,j)$.

Note $e_{ij}=e_{ji}e_{ij}$, as $e_{ji}e_{ij}=v_j^*v_iv_i^*v_j=v_j^*e_1v_j=v_j^*v_j=e_j$. Also, $e_ixe_j \in e_iMe_i=\mathbb{C}e_i$, so $e_ixe_j 1=x_{ij}e_i$ for some unique $x_{ij}\in\mathbb{C}$.

Let $x\in M$. $x=1\cdot x\cdot 1=(\sum_{i=1}^N e_i)\,x\,(\sum_{j=1}^N e_j)=\sum e_ixe_j=\sum e_ixe_{ji}e_{ij}=\sum (x_{ij}e_i)e_{ij}=\sum x_{ij}e_{ij}\in \operatorname{span}\{e_{ij}\}$.

$e_2 \overset{v_2}{\sim} e_1$, $v_2v_2^*=e_1$,

$E_{21}$ $E_{12}$

$v_2^*v_2=e_2$

So, any $x\in M$ can be uniquely written as $x=\sum x_{ij}e_{ij}$, where $x_{ij}$ is the unique scalar w/ $e_ixe_j 1=x_{ij}e_i$.

So, $\Phi: M\to M_n(\mathbb{C})$, $\Phi(\sum x_{ij}e_{ij})=\sum x_{ij}E_{ij}$. $\Phi$ is a $*$-morphism: it suffices to check $\Phi(e_{ij}^*)=\Phi(e_{ij})^*$ so

$\Phi(e_{ji})=E_{ij}^*=E_{ji}$. True!

$\Phi$ is multiplicative: $\Phi(e_{ij}e_{kl})\overset{?}{=}\Phi(e_{ij})\Phi(e_{kl})=E_{ij}\cdot E_{kl}=\delta_{jk}E_{il}$. We show $e_{ij}e_{kl}=\delta_{jk}E_{il}$ (and we are done).


So, $e_{ij}e_{kl}=v_i^*v_jv_k^*v_l=v_i^*\delta_{jk}e_1v_l=\delta_{jk}v_i^*v_l=\delta_{jk}e_{il}$.

$v_jv_k^*\{=0$ if $j\neq k$, as $v_k^*\{\in \operatorname{Range}(e_k)$

If $M$ is inf. dim., similarly let $(e_i)_{i\in I}$ be maximal as before. Fix $i_0\in I$, and define $e_j \overset{v_j}{\sim} e_{i_0}$, so $v_jv_j^*=e_{i_0}$, $v_j^*v_j=e_j$. Define $e_{ij}=v_i^*v_j$.

Let $H$ be a Hilbert space w/ ONB $(f_i)_{i\in I}$, and let $E_{ij}$ = rank 1 operator taking $e_j$ to $e_i$.

Define $\Phi(\sum x_{ij}e_{ij})=\sum x_{ij}E_{ij}$ for finite sums and extend s.o. Then $M \overset{\Phi}{\cong} B(H)$.

__Cor:__ If $M$ is a fin. dim. factor, then $M\cong M_n(\mathbb{C})$ ($M$ must be discrete).

__Cor:__ If $M$ is any fin. dim. $*$-algebra, then $M\cong M_{n_1}(\mathbb{C})\oplus\dots\oplus M_{n_k}(\mathbb{C})=\left\{\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_k\end{pmatrix} : A_i\in M_{n_i}(\mathbb{C})\right\}$.

Proof: Strong induction by $\dim Z(M)$: If $\dim Z(M)=1$, then $M$ is a factor, so $M = M_n(\mathbb{C})$.

If $\dim Z(M)>1$, then $\exists p \in Z(M)$ proj., $p \neq 0,1$. $M = \begin{pmatrix} \boxed{pMp} & 0 \\ 0 & \boxed{qMq} \end{pmatrix}$ w/ $q = 1-p$. $M = 1\cdot M \cdot 1 = (p+q)M(p+q) = pMp + \overset{0}{\overset{\shortparallel}{pMq}} + \overset{0}{\overset{\shortparallel}{qMp}} + qMq = pMp + qMq = M_p \oplus M_q$.

$\dim Z(M_p) < \dim Z(M)$ and $\dim Z(M_q) < \dim Z(M)$ (if $x \in Z(M_p)$, then it commutes w/ $M_q$ also, as $xq=0$, so $x \in Z(M)$). Done by induction!

Not equivalent to any subprojection; compare to finite sets

Def: $p \in P(M)$ is finite if $\forall q \in P(M)$, $q \leq p$ and $q \sim p \Rightarrow q = p$.

Examples: (i) $M = L^\infty([0,1], m)$, $p = \chi_E \neq 0$ ($m(E)>0$). If $q \leq p$, then $q = \chi_F$ w/ $F \subseteq E$. $q \sim p \Rightarrow q = p$, b/c $u^*u = uu^*$ in Abelian alg. So, all projs. are finite! Think about the trace: trace of any $\chi_E$ is finite.

(ii) $M = B(H)$. $E_{11}$ is finite, $E_{11}+E_{22}$ is finite, etc. $I$ is not finite unless $H$ is fin. dim.

If $p \in B(H)$ and $\dim p(H) < \infty$, then $p$ is finite: if $q < p$, then $q(H) \subsetneq p(H)$, hence $\dim q(H) < \dim p(H)$ and $q \not\sim p$.

(iii) $M = R = \overline{\bigotimes}^{\geq 0} M_2(\mathbb{C})$, any proj. is finite. If $q < p$, then $\tau(q) < \tau(p)$. so $q \not\sim p$ ($q \sim p \Rightarrow \tau(q) = \tau(p)$).

$$
\begin{tikzcd}[column sep=small]
 & & \text{Fin. dim. } \mathrm{I}_n = M_n(\mathbb{C}) & \\
 & \text{Discrete} \arrow[ur] \arrow[r] & \text{Inf. dim. } \mathrm{I}_\infty = B(H),\ \dim H = \infty & \\
M \text{ factor} \arrow[ur] \arrow[dr] & & & \text{all proj. finite } \mathrm{II}_1 \text{ (trace exists)} \\
 & \text{Continuous} \arrow[r] \arrow[dr] & \text{Has fin. proj.} \arrow[ur] \arrow[r] & \mathrm{II}_\infty \text{ (just } \mathrm{II}_1 \otimes B(H)) \\
 & & \text{No fin. proj. } \mathrm{III} \text{ (just } \mathrm{II}_1 \rtimes \mathbb{R}) &
\end{tikzcd}
$$