Today: Property (T) for groups

Next time: Algebras, manifolds

No class next week!

Kazdhan: • Russian mathematician, moved to Harvard

• In a paper in 1969 (70 years old) of 3 pages, he created waves in the math world by introducing property (T).

Margulis: • Russian mathematician, moved to Yale.

• Fields medalist

• Defined relative property (T)

Recap: • G countable group

• H Hilbert space

• $U \in B(H)$ unitary

• $\pi: G \to U(H)$, $g \mapsto \pi(g)$ w/ $\pi(g): H \to H$, $\xi \mapsto \pi(g)\xi$. π is a unitary group representation, so $\pi(gh) = \pi(g)\pi(h)$ and $\pi(g^{-1}) = \pi(g)^{-1}$.

• Matrix coefficients: $\forall \xi, \eta \in H$, $g \mapsto \langle \xi, \pi(g)\eta \rangle$ ← Note: conjugate to our usual matrix coeff.

• Trivial rep: $\pi(g) = \mathrm{Id}_H$, denoted 1_G.

• Left regular rep: $\lambda: G \to U(\ell^2(G))$, $g \mapsto \lambda(g)$. $\lambda(g)(f)(x) = f(g^{-1}x)$.

• Right regular rep: $\rho: G \to U(\ell^2(G))$. $\rho(g)(f)(x) = f(xg)$.

• Notation: (π, H)

Q: How can one compare two representations?

Weak containment (rep): (π, H), (ρ, K) two rep. of G.

$\pi \prec \rho$: all coeff. of π can be approximated by a finite sum of coeff. of ρ. i.e.

$\forall \varepsilon > 0$, $\forall \xi \in H$, $\forall$ finite set $F \subset G$, $\exists \eta_1, \ldots, \eta_n \in K$ s.t. $|\langle \xi, \pi(g)\xi \rangle - \sum_{i=1}^n \langle \eta_i, \rho(g)\eta_i \rangle| < \varepsilon$ $\forall g \in F$

Fell - Topology: (π, H) unitary rep. on G.

For $\varepsilon > 0$, $\xi \in H$, $F \subset G$ finite set, $V(\pi, \varepsilon, \xi, F) := \{(\rho, K) : \exists \eta \in K \text{ s.t. } |\langle \xi, \pi(g)\xi \rangle - \langle \eta, \rho(g)\eta \rangle| < \varepsilon \ \forall g \in F\}$ defines a subbasis of nbds of π for

$Rep(G)$.

So, $\pi \prec \rho \Rightarrow \rho$ is any nbd of π.

• $1_G \prec \rho \Rightarrow \rho$ is any nbd of 1_G. What does this look like?

$\|\xi\| = 1$ $\qquad\qquad$ $\|\eta\| = 1$

$U(1_G, \varepsilon, \xi, F) = \{(\rho, K) : \exists \eta \in K \text{ s.t. } |\langle \xi, \xi \rangle - \langle \eta, \pi(g)\eta \rangle| < \varepsilon \ \forall g \in F\}$ $\qquad$ $1 - \langle \eta, \pi(g)\eta \rangle \in \mathbb{C}$, so $1 - \mathrm{Re}\langle \eta, \pi(g)\eta \rangle = \frac{1}{2}\|\eta - \pi(g)\eta\|^2$

$= \{(\rho, K) : \exists \eta \in K \text{ s.t. } |1 - \langle \eta, \pi(g)\eta \rangle| < \varepsilon \ \forall g \in F\}$ $\qquad$ and $\langle \eta - \pi(g)\eta, \eta - \pi(g)\eta \rangle = 2(1 - \mathrm{Re}\langle \eta, \pi(g)\eta \rangle)$

$\Rightarrow U'(1, \varepsilon, F) = \{(\pi, K) : \exists \eta \in K, \|\eta\| = 1, \text{ s.t. } \sup_{g \in F} \|\eta - \pi(g)\eta\| < \varepsilon\}$ form a basis of nbds of 1_G.

In order for a rep. to be close to 1_G, it must have almost invariant vectors.

Def: Let (π, H) be a unitary rep.

(1) Given $F \subset G$ finite set, $\varepsilon > 0$, a vector η is (F, ε)-invariant if $\sup_{g \in F} \|\eta - \pi(g)\eta\| < \varepsilon$.

(2) It has almost invariant vectors if it has (F, ε)-invariant vectors $\forall F \subset G$ finite, $\forall \varepsilon > 0$.

Remark: $1_G \prec \pi \Leftrightarrow (\pi, H)$ has almost invariant vectors.

Def: (Property (T), or Kazhdan Prop.) A finitely generated group G has Property (T) if every unitary rep. that has an almost invariant vector has an invariant vector (nonzero).

$(\pi, H), (\rho, K)$ rep. $\rho \subset \pi$ if $K \subset H$ and $\pi|_K = \rho$.

Remark: In terms of Fell topology, G having Property (T) $\iff 1_G \prec \pi \Rightarrow 1_G \subset \pi$.

$\xi \in H$ s.t. $\pi(g)\xi = \xi$. $\mathbb{C}\xi \subset H$, $\pi(g)(c\xi) = c\xi$, $\pi|_{\mathbb{C}\xi} = 1_G$.

So, Property (T) means trivial rep. is isolated.

This is called a "rigidity" property.

Examples: (1) Every finite group has Property (T). Not obvious!

(2) $SL_n(\mathbb{Z})$, $SL_n(\mathbb{R})$ have Property (T) for $n \geq 3$.

(3) "Almost" every group has Property (T).

Nonexamples: (1) Abelian groups

(2) $\mathbb{F}_n$, $n \geq 2$.