__Lemma:__ If $T \in B(H)$, K Hilbert subspace of H, then T, T^* leave K invariant $\iff$ proj$_K$ commutes with T.

need unital b/c M' always contains I

__Bicommutant Thm:__ If $M \subset B(H)$ is a unital $*$-alg., then $M = \overline{M}^{s.o.} \iff M = M''$.

$\Leftarrow$ is clear b/c any X' is s.o. closed $\forall X \in B(H)$.

We show: $M = \overline{M}^{s.o.} \Rightarrow M = M''$. $M \subset M''$ always holds, so we just need to worry about $M'' \subset M$. So, let $x \in M''$. To show $x \in M$, we must use $M = \overline{M}^{s.o.}$, so we will probably have an easier time showing $x \in \overline{M}^{s.o.}$. A basis of neighborhoods of x in the s.o. topology is given by $V(x, \varepsilon, \xi_1, \dots, \xi_n) = \{y \in B(H) : \|y\xi_i - x\xi_i\| < \varepsilon \ \forall i = 1, \dots, n\}$. We must show: $\forall \varepsilon > 0$, $\forall \xi_1, \dots, \xi_n \in H$, $\exists y \in M$ s.t. $\|y\xi_i - x\xi_i\| < \varepsilon \ \forall i$.

__Consider the case $n=1$:__ $\xi_1 = \xi$. $\varepsilon, \xi, x \in M''$ are given, want $y \in M$ w/ $\|y\xi - x\xi\| < \varepsilon$. Let $K = \overline{M\xi} = \overline{\{y\xi : y \in M\}} \subset H$. Must show $x\xi \in K$.

Note that K is invariant to all operators in M (and in $M^* = M$) ($T \in M \Rightarrow Ty\xi \in M$ as $Ty \in M \ \forall y \in M$). So, P_K commutes with M, hence $P_K \in M'$. But $x \in M'' = (M')'$ and $P_K \in M'$, so $xP_K = P_K x$. $\Rightarrow xP_K\xi = P_K x\xi \Rightarrow x\xi = P_K(x\xi) \Rightarrow x\xi \in K$. Done!

__The general case:__ Given $\varepsilon > 0$, $\xi_1, \dots, \xi_n \in H$, $x \in M''$. Want: $y \in M$, $\|y\xi_i - x\xi_i\| < \varepsilon \ \forall i$. Define $H^n = \overbrace{H \oplus \cdots \oplus H}^{n \text{ times}} = \{(v_1, \dots, v_n) \in H^n\}$ w/

$\langle (v_1, \dots, v_n), (w_1, \dots, w_n) \rangle = \sum_i \langle v_i, w_i \rangle$. Note $B(H^n) \cong M_n(B(H))$. $\begin{matrix} H \\ \oplus \\ H \end{matrix} \xrightarrow{T} \begin{matrix} H \\ \oplus \\ H \end{matrix}$, $T = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$. Let $\xi = (\xi_1, \dots, \xi_n) \in H^n$, and let

$\tilde{M} = \left\{ \begin{pmatrix} y & & 0 \\ & \ddots & \\ 0 & & y \end{pmatrix} : y \in M \right\} \subset M_n(B(H)) = B(H^n)$. In other words, we identify $y \in M$ to an operator $\tilde{y} \in \tilde{M}$ w/ $\tilde{y}$ acting on H^n:

$\tilde{y}(v_1, \dots, v_n) = (yv_1, \dots, yv_n)$. $\tilde{M} = \{\tilde{y} : y \in M\} \subset B(H^n)$ (this is still s.o. closed).

Exercise

We show: $x \in M'' \Rightarrow \tilde{x} \in \tilde{M}''$. $\tilde{M}' = M_n(M')$ (in $B(H^n)$). So, $\tilde{M}'' = (M_n(M'))' = \widetilde{M''} = \left\{ \begin{pmatrix} x & & 0 \\ & \ddots & \\ 0 & & x \end{pmatrix} : x \in M'' \right\}$. Now, use part 1 of the proof

(for one vector) for $\tilde{M}$, $\varepsilon > 0$, $\xi = (\xi_1, \dots, \xi_n)$: $\tilde{x} \in \tilde{M}'' \Rightarrow \exists \tilde{y} \in \tilde{M}$, $\|\tilde{y}\xi - \tilde{x}\xi\| < \varepsilon$. So, $\tilde{y} = (y, \dots, y)$, $y \in M$, and $\|(y\xi_1, \dots, y\xi_n) - (x\xi_1, \dots, x\xi_n)\| < \varepsilon \Rightarrow$

$\sum_i \|y\xi_i - x\xi_i\|^2 < \varepsilon^2 \Rightarrow \|y\xi_i - x\xi_i\| < \varepsilon \ \forall i$. ∎

__Notation:__ If $\xi, \eta \in H$, denote by $\omega_{\xi,\eta}$ the map $\omega : B(H) \to \mathbb{C}$, $\omega_{\xi,\eta}(x) = \langle x\xi, \eta \rangle \in \mathbb{C}$.

Thm: Let $\omega : B(H) \to \mathbb{C}$ be linear. TFAE:

(1) ω is w.o. cont.

(2) ω is s.o. cont.

(3) $\omega = \sum_{i=1}^{n} \omega_{\xi_i, \eta_i}$ (for some $\xi_i, \eta_i \in H$)

Proof: (1) ⇒ (2): If $x_i \xrightarrow{s.o.} x$, then $x_i \xrightarrow{w.o.} x$, so by (1), $\omega(x_i) \to \omega(x)$.

(3) ⇒ (1) is clear.

(2) ⇒ (3): Let $\omega : B(H) \to \mathbb{C}$ be s.o. cont. ⇒ ω is s.o. cont. at 0. So, $\omega^{-1}(\{z \in \mathbb{C} : |z| \le 1\})$ must contain an open nbd of 0 in $(B(H), s.o.)$.

So, it must contain $\{x \in B(H) : \|x\xi_i\| < \varepsilon \ \forall i\}$ for some $\varepsilon > 0$, $\xi_i \in H$. If $\|x\xi_i\| < \varepsilon \ \forall i$ for some $x \in B(H)$, then $|\omega(x)| \le 1$.

Let x be arbitrary in $B(H)$. $\|x\xi_i\|$ may be $\ge \varepsilon$, but there is $k = k(x) > 0$ s.t. $\frac{1}{k}\|x\xi_i\| < \varepsilon \ \forall i$, i.e. $\|(\frac{1}{k}x)\xi_i\| < \varepsilon \ \forall i$ ⇒ $|\omega(\frac{1}{k}x)| \le 1$

$|\omega(x)| \le k = \varepsilon^{-1} \sum \|x\xi_i\| = \sum \|x\eta_i\|$ w/ $\eta_i := \varepsilon^{-1}\xi_i$.

will finish next time!