Thm: For $\omega : B(H) \to \mathbb{C}$ linear, TFAE:

(1) ω is w.o. cont.

(2) ω is s.o. cont.

(3) $\omega = \omega_{\xi_1,\eta_1} + \cdots + \omega_{\xi_n,\eta_n}$ $(\xi_i, \eta_i \in H)$ w/ $\omega_{\xi,\eta} = \langle x\xi, \eta \rangle$.

Remark: It is known that if 2 locally convex topologies have the same linear cont. functionals, then they have the same closed convex sets.

So, it follows that $K \subset B(H)$ convex is s.o. closed iff it is w.o. closed. Thus, we could have defined v.N. alg. as w.o. closed instead of s.o. closed.

Proof: (3) $\Rightarrow$ (1) and (1) $\Rightarrow$ (2) are clear. Think $|\omega(x)| \leq k \cdot \|x\|$

(2) $\Rightarrow$ (3): s.o. topology is given by seminorms $x \mapsto \|x\xi_1\| + \cdots + \|x\xi_n\|$ $(\xi_i \in H)$. Thus, if ω is cont. s.o., then $\exists \xi_1, \ldots, \xi_n \in H$ s.t.

$|\omega(x)| \leq \|x\xi_1\| + \cdots + \|x\xi_n\|$ $\forall x \in B(H)$. $a_1 + \cdots + a_n \leq (1^2 + \cdots + 1^2)^{1/2} (a_1^2 + \cdots + a_n^2)^{1/2}$

by Cauchy-Schwarz

Let $\varphi : V \to \mathbb{C}$, $V = \{(x\xi_1, \ldots, x\xi_n) : x \in B(H)\} \subseteq H^n$, $\varphi(x\xi_1, \ldots, x\xi_n) = \omega(x)$ $\forall x \in B(H)$. $|\omega(x)| \leq \|x\xi_1\| + \cdots + \|x\xi_n\| \leq \sqrt{n} (\|x\xi_1\|^2 + \cdots + \|x\xi_n\|^2)^{1/2}$, so φ is bounded and linear on V. Hence, it can be continuously extended to $\bar{V} \subseteq H^n$. $\Rightarrow \varphi \in V^*$, so by Riesz, $\exists (\eta_1, \ldots, \eta_n) \in \bar{V} \subseteq H^n$ s.t.

$\underbrace{\varphi(x\xi_1, \ldots, x\xi_n)}_{=\,\omega(x)} = \langle (x\xi_1, \ldots, x\xi_n), (\eta_1, \ldots, \eta_n) \rangle = \sum_{i=1}^n \langle x\xi_i, \eta_i \rangle$. Done!

Thm: Let $M \subset B(H)$ be a v.N. alg. If $\omega : M \to \mathbb{C}$ is linear and s.o. cont., then $\exists \xi_i, \eta_i \in H$ s.t. $\omega = \sum_{i=1}^n \omega_{\xi_i,\eta_i}$.

Proof: As before, we can argue that $|\omega(x)| \leq p(x)$ $\forall x \in M$, where $p(x) = \|x\xi_1\| + \cdots + \|x\xi_n\|$ for some $\xi_i \in H$. By Hahn-Banach, since p is defined on all $B(H)$, we can extend ω to $\tilde{\omega}$ on $B(H)$ w/ $|\tilde{\omega}(x)| \leq p(x)$ $\forall x \in B(H)$. So, by the prev. result, $\tilde{\omega} = \sum \omega_{\xi_i,\eta_i}$. Done!

Review seminorms and functional analysis results

Geometry of Projections (Section 2.4)

Recall: If $p,q \in B(H)$ projections, we say $p \leq q$ if $p(H) \subseteq q(H)$.

Thm: For $p,q \in B(H)$ projections, TFAE:

(i) $p \leq q$ (iv) $\langle p\xi,\xi\rangle \leq \langle q\xi,\xi\rangle \ \forall \xi \in H$

(ii) $pq = p$ (v) $q-p$ is positive

(iii) $\|p\xi\| \leq \|q\xi\| \ \forall \xi \in H$

Remark: All are easy to check. (iv)$\Leftrightarrow$(iii): $\langle p\xi,\xi\rangle \leq \langle q\xi,\xi\rangle \Leftrightarrow \langle p^2\xi,\xi\rangle \leq \langle q^2\xi,\xi\rangle \Leftrightarrow \langle p\xi,p\xi\rangle \leq \langle q\xi,q\xi\rangle \Leftrightarrow \|p\xi\|^2 \leq \|q\xi\|^2$.

Def: For $p_i \in B(H)$ projections ($i \in I$), define

Think $Q = A \cap B$, $Q \subset A, Q \subset B$, and $R \subset A, B \Rightarrow R \subset Q$

$\bigwedge_{i\in I} p_i = \mathrm{proj}_{\bigcap p_i(H)} =$ the unique proj. $q \in B(H)$ w/ $\begin{cases} q \leq p_i & \forall i \in I \\ r \leq p_i \text{ proj. } \forall i \in I \Rightarrow r \leq q \end{cases}$, and

$\bigvee_{i\in I} p_i = \mathrm{proj}_{\overline{\mathrm{span}\{p_i(H) : i \in I\}}^{\|\cdot\|}} =$ the unique proj. $s \in B(H)$ w/ $\begin{cases} p_i \leq s & \forall i \in I \\ p_i \leq t \text{ proj. } \forall i \in I \Rightarrow s \leq t. \end{cases}$

Complete b/c we can always do $\wedge, \vee$.

We say that $(\mathcal{P}(B(H)), \wedge, \vee, \leq)$ is a complete lattice.

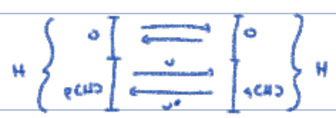
Remark: Let $M \subset B(H)$ be a v.N. alg. If $p,q \in \mathcal{P}(M)$, then $p \wedge q = \lim^{s.o.}_{n\to\infty} (pq)^n \in M$ and $p \vee q = 1-(1-p)\wedge(1-q) \in M$.

If $p = P_K$, $q = P_L$, $1-p = P_{K^\perp}$, $1-q = P_{L^\perp}$, $\Rightarrow (1-p)\wedge(1-q) = P_{K^\perp \cap L^\perp} = P_{(K+L)^\perp} = 1 - P_{K+L} = 1 - P_K \vee P_L$.

It can also be shown: if $(p_i)_{i\in I} \in M$ proj., then $\bigwedge p_i, \bigvee p_i \in M$. See book!

So, $(\mathcal{P}(M), \wedge, \vee, \leq)$ is a complete lattice.

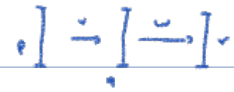
Def: On $\mathcal{P}(M)$, define $\sim$ by $p \sim q$ if $\exists v \in M$ (p.i.) w/ $vv^* = q$, $v^*v = p$.



Prop: This is an equivalence relation on $\mathcal{P}(M)$.

Proof: Reflexive: $p \sim p$ via p.

__Symmetric:__ $p\sim q$ via $v \Rightarrow q\sim p$ via v^*.



__Transitive:__ if $p\sim q$ via v, i.e. $vv^*=q$ and $v^*v=p$, and $q\sim r$ via w, i.e. $ww^*=r$ and $w^*w=q$, then $p\sim r$ via wv.

$wq=ww^*w=w$

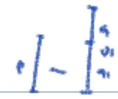
Indeed, $(wv)(wv)^*=wvv^*w^*=wqw^*=ww^*=r$. Similarly, $(wv)^*(wv)=p$.

Note: w, wq are same on $q(H)$ (as $q^*=q$) and are the same (0) on $q(H)^\perp$.

Or: $wq\xi=w\xi$? $\xi\in q(H)\Rightarrow q\xi=\xi$ and $w\xi=\xi$

$\xi\in q(H)^\perp \Rightarrow q\xi=0$ and $w\xi=0$

__Def:__ For $p,q\in P(M)$, we write $p\prec q$ if $\exists q_1\le q$, $q_1\in P(M)$, s.t. $p\sim q_1$.



A better notation would be $p\precsim q$, but for reasons of being lazy, we denote it by $p\prec q$.

__Will see:__ $p\prec q$, $q\prec p\Rightarrow p\sim q$.