<u>Recall:</u> A fin. gen. group G has Property (T) if every unitary rep of G that has almost invariant vectors has a nonzero invariant vector.

<u>Groups w/ property (T):</u> Finite groups, compact top. groups, $SL(n,\mathbb{R})$ & $SL(n,\mathbb{Z})$ w/ $n \geq 3$, simple real Lie groups

<u>Groups w/o property (T):</u> $\mathbb{Z}, \mathbb{R}$, $SL(2,\mathbb{Z}), SL(2,\mathbb{R})$, p-adic rationals, nontrivial free groups, free Abelian groups

<u>Thm:</u> (Kazhdan, 1967) $SL(n,\mathbb{Z})$ has (T) for $n \geq 3$.

<u>Proof:</u> (Burger/Shalom)

We will just look at $n=3$. $SL(3,\mathbb{Z}) = \{A \in M_3(\mathbb{Z}) : \det A = 1\}$ contains some interesting subgroup:

$G = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{pmatrix} : a,b,c,d,e,f \in \mathbb{Z}, \begin{pmatrix} a & b \\ d & e \end{pmatrix} \in SL(2,\mathbb{Z}) \right\}$. G contains 2 subgroups: $SL(2,\mathbb{Z})$ as $\left\{ \begin{pmatrix} a & b & 0 \\ d & e & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}$ and $\mathbb{Z}^2$ as $\left\{ \begin{pmatrix} 1 & 0 & c \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix} \right\}$.

What is the relation between their generators? For $g \in SL(2,\mathbb{Z}), S \in \mathbb{Z}^2$, $gSg^{-1} = (gS)$ (matrix g mult. by vector S).

We write $G = \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$, cross product of action of $SL(2,\mathbb{Z})$ on $\mathbb{Z}^2$.

<u>Note:</u> There are several other ways to embed G in $SL(3,\mathbb{Z})$. $\mathbb{Z}^2 \to \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ * & * & 1 \end{pmatrix}, \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 1 & * & * \end{pmatrix}, \ldots$ 6 ways total.

Their union is $SL(3,\mathbb{Z})$, and the union of the corresponding $\mathbb{Z}^2$'s is still $SL(3,\mathbb{Z})$.

We prove:

← H is like an anchor

<u>Thm:</u> $\underset{= H}{\mathbb{Z}^2} \subset \underbrace{\mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})}_{= G}$ has <u>relative property (T)</u>, meaning any unitary rep. of G with almost invariant vector(s) admits an invariant vector for H.

If we prove this, it follows that $SL(3,\mathbb{Z})$ has (T).

More specifically, $\varepsilon = 1/10$, $Q = \left\{ \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}, \begin{pmatrix} \pm 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ \pm 1 \end{pmatrix} \right\}$ (viewed as elements of G).

We show: if $\pi : G \to B(\mathcal{H})$ is a unitary rep. w/ $\xi \in \mathcal{H}, \xi \neq 0$, $\|\pi(g)\xi - \xi\| < \varepsilon \ \forall g \in Q$, then $\exists \eta \neq 0$ in $\mathcal{H}$ w/ $\pi(g)\eta = \eta \ \forall g \in H$.

$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $C^*(u,v) = \overline{\{p(u,v) : p \in \mathbb{C}[x,y]\}}^{\|\ \|}$

$\mathbb{Z}^2 \subset G$, so $\mathbb{Z}^2$ is mapped by π into $\pi(\mathbb{Z}^2)$ (unitaries on $\mathcal{H}$). So, π extends from $\mathbb{Z}^2$ to $C^*(\mathbb{Z}^2) = C^*(u,v) = C(\mathbb{T}^2)$.

In fact, π extends to $v.N.(\hat{\mathbb{Z}}^2) = L^\infty(\mathbb{T}^2, m)$. So, $\pi(\chi_E)$ makes sense for $E \subset \mathbb{T}^2$.

Assume, by contradiction, that π has no (nonzero) invariant vectors. η invariant for $\pi|_{\mathbb{Z}^2}$ means $\pi(u)\eta = \eta$ and $\pi(v)\eta = \eta$.

Equivalently, $\pi(T)\eta = \eta \quad \forall T \in v.N.(\mathbb{Z}^2) = L^\infty(\mathbb{T}^2)$.

or $\chi_{\{(1,1)\}}(T)$ here

Recall: If T is a normal operator, then $E_1 = \{\eta : T\eta = \eta\} = \chi_{\{1\}}(T)$.

So, no invariant vectors means $\pi(\chi_{\{(1,1)\}}) = 0$.

Plan: We can identify $\mathbb{T}^2$ to $(-\frac{1}{2}, \frac{1}{2}]^2$ via $(x, y) \in (-\frac{1}{2}, \frac{1}{2}]^2 \mapsto (e^{2\pi i x}, e^{2\pi i y}) \in \mathbb{T}^2$. Via this identification, we can translate all information

$f = \chi_{\{(1,1)\}} \Rightarrow \; = \mu_{\xi,\xi}(\{(1,1)\})$

from our problem to information regarding $(-\frac{1}{2}, \frac{1}{2}]^2$, the action of $\mathbb{Z}^2$ on it, and $\mu = \mu_{\xi,\xi}$ on $(-\frac{1}{2}, \frac{1}{2}]^2$. Recall: $\langle f(T)\xi, \xi \rangle = \int f \, d\mu_{\xi,\xi}$.

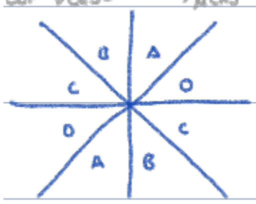
No invariant vectors for $\pi \longrightarrow \mu$ has no mass at $(0,0)$

$g = \binom{1}{0}$

ξ is ε-invariant to $\left\{ \binom{\pm 1}{0}, \binom{0}{\pm 1} \right\} \longrightarrow \mu(X) \geq 1 - \varepsilon^2$ w/ $X = (-\frac{1}{4}, \frac{1}{4}]^2$ $\qquad \varepsilon^2 \geq \|\pi(g)\xi - \xi\|^2 = \int |e^{2\pi i x} - 1|^2 \, d\mu_{\xi,\xi} \geq 2$ for $|x| > \frac{1}{4}$

ξ is ε-invariant to $\left\{ \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \right\} \longrightarrow |\mu(gE) - \mu(E)| \leq \varepsilon \quad \forall E \subset (-\frac{1}{2}, \frac{1}{2}]^2$, $\forall g$ one of the 4 generators $\quad g \in SL(2, \mathbb{Z}),\ s \in \mathbb{Z}^2,\ gsg^{-1} = (gs)$

Let $\nu(E) = \mu(E \cap X) / \mu(X) \quad \forall E \subset \mathbb{R}^2 \setminus \{0\}$ Borel. ν is a prob. measure on $\mathbb{R}^2 \setminus \{0\}$, $|\nu(gE) - \nu(E)| < \frac{1}{4} \quad \forall g \in \left\{ \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \right\}$.



$g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $g(A \cup B) \subset A$, so $\nu(g(A \cup B)) - \nu(A \cup B) < \frac{1}{4} \Rightarrow \nu(B) < \frac{1}{4}$.

<u>Important idea:</u> change problem w/ operators to problem w/ measures.