Perturbations of Operator Algebras

Oral Exam Presentation

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Erik Christensen

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- We focus on the article in Math Annalen titled "Subalgebras of a Finite Algebra."

General Setting (von Neumann algebras)

- For a Hilbert space $\mathcal{H}$, a von Neumann algebra L is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ that is closed in the strong operator topology.

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  - The closure condition is equivalent to $L = L''$, where

    $$L' = \{x \in \mathcal{B}(\mathcal{H}) : xy = yx \ \forall y \in L\}$$

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    is the commutant of L in $\mathcal{B}(\mathcal{H})$ (this is von Neumann's Bicommutant Theorem).
- On $\mathcal{B}(\mathcal{H})$, we have the usual norm $||T|| = \sup_{||x|| \leq 1} ||Tx||$.

General Setting (trace)

- We require that L has a trace $\tau : L \to \mathbb{C}$ (so τ is linear and $\tau(xy) = \tau(yx)\ \forall x, y \in L$) that is:
  - positive: $\tau(x^*x) \geq 0\ \forall x \in L$
  - faithful: $\tau(x^*x) = 0 \Rightarrow x = 0$
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  - faithful: $\tau(x^*x) = 0 \Rightarrow x = 0$
  - normal: weakly continuous, $\tau(1) = 1$
- A von Neumann algebra with a trace satisfying these properties is called <u>finite</u> (not to be confused with finite dimensional).

General Setting (in finite dimensions)

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- So, a helpful example to think of is

$$L = M_{k_1}(\mathbb{C}) \oplus M_{k_2}(\mathbb{C}) \oplus \cdots \oplus M_{k_n}(\mathbb{C})$$

with trace

$$\tau = c_1 \tau_{M_{k_1}} + \cdots + c_n \tau_{M_{k_n}},$$

where $\tau_{M_k} = \frac{1}{k} \operatorname{Tr}_{M_k}$ and $c_1 + \cdots + c_n = 1$.

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  where $\tau_{M_k} = \frac{1}{k} \operatorname{Tr}_{M_k}$ and $c_1 + \cdots + c_n = 1$.
- These can be thought of as tuples of matrices $(A_{k_1}, \ldots, A_{k_n})$ or block matrices

$$\begin{pmatrix} A_{k_1} & & 0 \\ & \ddots & \\ 0 & & A_{k_n} \end{pmatrix}$$

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- If L is finite dimensional (in which case $L \cong M_k(\mathbb{C})$), we say L is a factor of type I_k.
- If L is infinite dimensional, we say L is a factor of type II_1.

General Setting (what it means to be "close")

In these papers, Christensen studies subalgebras (or rather, pairs of subalgebras) $M, N \subset L$ that are "close" in the sense that

$$\forall x \in M_1, \exists y \in N \text{ such that } ||x - y||_2 < \delta \qquad (*)$$

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- M_1 denotes the unit ball of M.
- $|| \ ||_2$ denotes the norm induced by the trace: $||x||_2 = \tau(x^*x)^{1/2}$.
- $\overline{M}^{|| \ ||_2}$ is the GNS construction and is denoted by $L^2(M, \tau)$.

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We write $M \overset{\delta}{\subset} N$ when $(*)$ occurs.

Important Remark

For some of these proofs, Christensen introduces what will later be known as Jones' basic construction, from Jones' fields medal paper (83).

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Theorem I

We prove two results, then combine them. Recall that (L, τ) is a finite von Neumann algebra.

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Theorem (I)

If $M, N \subset L$ with M finite dimensional, N a II$_1$ factor, and $M \overset{\delta}{\subset} N$ for some $\delta < 1/\sqrt{2}$, then there exist $\Phi : M \to N$, Φ an isomorphism from M to a von Neumann subalgebra $\Phi(M)$ of N, and a fixed constant c such that for all $x \in M_1$

$$||\Phi(x) - x||_2 \leq c\delta^{1/2}$$

($c = 1050$ should work).

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Theorem II

Theorem (II)

If $M \subset L$ is a von Neumann subalgebra of L and $\Phi : M \to L$ is a $*$-homomorphism such that $||\Phi(x) - x||_2 \leq t$ for all $x \in M_1$, for some $0 < t < 1$, then for all $x \in M$

$$\Phi(x)q = v^*xv,$$

where $v \in L$ is a partial isometry with

- $q := v^*v \in \Phi(M)'$
- $r := vv^* \in M'$
- $||1 - v||_2 \leq 2t$, $||1 - r||_2 \leq t$, $||1 - q||_2 \leq t$.

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v being a partial isometry means that v^*v, vv^* are projections.

Theorem II′

We also prove a stronger version of II when M is a I_k factor (so $M \cong M_k(\mathbb{C})$).

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Theorem (II′)

Under the hypotheses of II, if M is a I_k factor, then there is a unitary u in L such that $\Phi(x) = u^*xu$ for all $x \in M$ and $||1 - u||_2 \leq 3t$.

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Under the hypotheses of II, if M is a I_k factor, then there is a unitary u in L such that $\Phi(x) = u^* x u$ for all $x \in M$ and $||1 - u||_2 \leq 3t$.

u being unitary means that $u^* u = u u^* = 1$. We write $\mathcal{U}(L)$ for the set of unitaries in L. Any element of L can be written as a linear combination of unitaries.

Theorem III

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Theorem (III)

If $M \overset{\delta}{\subset} N$ with M I$_k$, N II$_1$, and $\delta < 10^{-6}$, then there is a unitary u in L such that $u^*Mu \subset N$ and $||1-u||_2 \leq 450\delta^{1/2}$.

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Proof idea: I gives us the map Φ and II′ tells us that $\Phi(x) = u^*xu$.

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Theorem II

Theorem (II)

If $M \subset L$ *is a von Neumann subalgebra of* L *and* $\Phi : M \to L$ *is a* $*$*-homomorphism such that* $||\Phi(x) - x||_2 \leq t$ *for all* $x \in M_1$, *for some* $0 < t < 1$, *then for all* $x \in M$

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where $v \in L$ *is a partial isometry with*

- $q := v^*v \in \Phi(M)'$
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Proof of II (step 1)

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- Step 1: Come up with an intertwiner: $k \in L$ with $k\Phi(u) = uk$.

$$
\begin{aligned}
&||\Phi(x) - x||_2 \leq t \ \forall x \in M_1 \\
\Rightarrow &||\Phi(u) - u||_2 \leq t \ \forall u \in \mathcal{U}(M) \\
\Rightarrow &||u^*\Phi(u) - 1||_2 \leq t \ \forall u \in \mathcal{U}(M) \ (||uz||_2 = ||zu||_2 = ||z||_2)
\end{aligned}
$$

Proof of II (step 1)

Consider

$$K = \overline{\left\{ \sum_{i=1}^{n} c_i u_i^* \Phi(u_i) : n \geq 1, u_i \in \mathcal{U}(M), c_i > 0, \sum_{i=1}^{n} c_i = 1 \right\}}^{\|\ \|_2}.$$

K is a convex closed set in the Hilbert space $L^2(L, \tau)$, so it has a unique element k of minimal norm $||\ ||_2$.

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Proposition

If $C \subset L$ is a bounded set (in regular norm $\|\ \|$), then $\overline{C}^{\text{s.o.}}$ (which is part of L, as $L = \overline{L}^{\text{s.o.}}$) is the same as $\overline{C}^{\|\ \|_2}$ (in $L^2(L,\tau)$).

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Proposition

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Since

$$||c_1 u_1^* \Phi(u_1) + \cdots + c_n u_n^* \Phi(u_n)|| \leq c_1 + \cdots + c_n = 1,$$

it follows that $K \subset L$. In particular, $k \in L$.

Proof of II (step 1)

Note that for all $u \in \mathcal{U}(M)$, we have $u^* K \Phi(u) \subset K$. Indeed, for all $u_0 \in \mathcal{U}(M)$:

$$u^*(u_0^* \Phi(u_0))\Phi(u) = (u_0 u)^* \Phi(u_0 u) \in K.$$

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In particular, $u^*k\Phi(u) \in K$. Also, $||u^*k\Phi(u)||_2 = ||k||_2$.

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In particular, $u^* k \Phi(u) \in K$. Also, $||u^* k \Phi(u)||_2 = ||k||_2$.

By uniqueness, it follows that $u^* k \Phi(u) = k$, or $k\Phi(u) = uk$.

Proof of II (step 1)

$k\Phi(u) = uk$ doesn't say anything if $k = 0$. However,

$$||u^*\Phi(u) - 1||_2 \leq t < 1 \Rightarrow ||k_0 - 1||_2 \leq t < 1$$

for all $k_0 \in K$, so $||k - 1||_2 \leq t < 1$. Hence, $k \neq 0$.

Proof of II (step 2)

- Step 2: From k, we collect its "unitary" part.

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Consider the polar decomposition $k = va$. Here

$$a = |k| = (k^*k)^{1/2}$$

and v is a partial isometry, with

$$\ker k = \ker v = \ker a.$$

Since v is a partial isometry, $q := v^*v$ and $r := vv^*$ are projections.

Proof of II (step 2)

We have $k = va$. It is known that v, a are still in L, since $k \in L$ and L is a von Neumann algebra.

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We have $k = va$. It is known that v, a are still in L, since $k \in L$ and L is a von Neumann algebra.
Also,

$$q = v^*v = s_r(k) = s(k^*k)$$

and

$$r = vv^* = s_\ell(k) = s(kk^*).$$

s_ℓ, s_r are the projections onto $\overline{\mathrm{Im}\, k}$, $(\ker k)^\perp$ respectively and are called the left and right support. $s = s_\ell = s_r$ for self-adjoint operators and is just called the support. s, s_ℓ, s_r are also in L.

Proof of II (step 2)

From $||k - 1||_2 \leq t$, it can be shown that

$$||1 - v||_2 \leq 2t, \; ||1 - q||_2 \leq t, \; ||1 - r||_2 \leq t$$

(this follows from a technical lemma in the paper, which we skip).

Proof of II (step 2)

From Step 1, we know $k\Phi(u) = uk$ for all $u \in \mathcal{U}(M)$. Taking adjoints gives $\Phi(u^*)k^* = k^*u^*$. Replacing u^* with u gives $\Phi(u)k^* = k^*u$. So, for all $u \in \mathcal{U}(M)$

$$\begin{cases} k\Phi(u) = uk \\ \Phi(u)k^* = k^*u \end{cases}$$

Proof of II (step 2)

$$\begin{cases} k\Phi(u) = uk \\ \Phi(u)k^* = k^*u \end{cases}$$

$$\Rightarrow k^*(k\Phi(u)) = k^*(uk) = (k^*u)k = (\Phi(u)k^*)k$$

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$$\Rightarrow k^*(k\Phi(u)) = k^*(uk) = (k^*u)k = (\Phi(u)k^*)k$$

This shows that k^*k commutes with $\Phi(u)$ for all $u \in \mathcal{U}(M)$, and thus $k^*k \in \Phi(M)'$. So, $a = (k^*k)^{1/2}$ and $q = s(k^*k)$ are in $\Phi(M)'$ also.

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$$\begin{cases} k\Phi(u) = uk \\ \Phi(u)k^* = k^*u \end{cases}$$

$$\Rightarrow (uk)k^* = (k\Phi(u))k^* = k(\Phi(u)k^*) = k(k^*u)$$

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This shows that kk^* commutes with u for all $u \in \mathcal{U}(M)$, and thus $kk^* \in M'$. So, $r = s(kk^*)$ is in M' also.

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&\Rightarrow \Phi(u)qa = v^*uva \quad (q, a \in \Phi(M)') \\
&\Rightarrow \Phi(u)q = v^*uv \quad (\mathcal{H} = \overline{a(\mathcal{H})} \oplus \overline{a(\mathcal{H})}^{\perp})
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\end{aligned}$$

It follows that $\Phi(x)q = v^*xv$ for all $x \in M$. QED.

Corollary

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Let (M, τ) be a finite factor. Let $\Phi : M \to M$ be a $*$-homomorphism such that $||\Phi(x) - x||_2 \leq t < 1$ for all $x \in M_1$ (for $0 < t < 1$). Then there is a unitary $u \in M$ with $||1 - u||_2 \leq 2t$ and $\Phi(x) = u^* x u$ for all $x \in M$.

- From II, $\Phi(x)q = v^* x v$ with $q = v^* v$ and $r = vv^*$.

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- From II, $\Phi(x)q = v^*xv$ with $q = v^*v$ and $r = vv^*$.
- $r \in M \cap M' = Z(M) = \mathbb{C} \cdot 1$.

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- r is a nonzero projection, so $r = 1$ (r is nonzero since $||1 - r||_2 < 1$).

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- Since q is a projection and τ is faithful, we get $q = 1$.

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- From II, $\Phi(x)q = v^* x v$ with $q = v^* v$ and $r = vv^*$.
- $r \in M \cap M' = Z(M) = \mathbb{C} \cdot 1$.
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- $v^* v = vv^* = 1$ implies v is a unitary, and $\Phi(x) = v^* x v$.

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- Since q is a projection and τ is faithful, we get $q = 1$.
- $v^* v = vv^* = 1$ implies v is a unitary, and $\Phi(x) = v^* x v$.
- We know $||1 - v||_2 \leq 2t$ from II.

Theorem II′

Theorem (II′)

Under the hypotheses of II, if M is a I_k factor, then there is a unitary u in L such that $\Phi(x) = u^*xu$ for all $x \in M$ and $||1 - u||_2 \leq 3t$.

Proof of II′ (idea)

Idea

From II, there exists $v \in L$ such that $q = v^*v \in \Phi(M)'$, $r = vv^* \in M'$, and $\Phi(x)q = v^*xv$ for all $x \in M$.

Proof of II′ (idea)

Idea

From II, there exists $v \in L$ such that $q = v^*v \in \Phi(M)'$, $r = vv^* \in M'$, and $\Phi(x)q = v^*xv$ for all $x \in M$.

The map $z \mapsto v^*zv$ sends Mr to $\Phi(M)q$:

$$xr \mapsto v^*xrv = v^*xvv^*v = \Phi(x)q^2 = \Phi(x)q.$$

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We extend this to a map $z \mapsto u^*zu$ ($u \in \mathcal{U}(M)$) that takes M to $\Phi(M)$. First, we get another partial isometry v_1 so that $z \mapsto v_1^*zv_1$ sends $M(1-r)$ to $\Phi(M)(1-q)$, then we let $u = v + v_1$.

Proof of II′ (picture)

$$\begin{tikzcd}
 & Mr \arrow[r, "z \mapsto v^* z v"] & \Phi(M)q \arrow[dr, dash] & \\
M \arrow[ur, dash] \arrow[dr, dash] & & & \Phi(M) \\
 & M(1-r) \arrow[r, "z \mapsto v_1^* z v_1"] & \Phi(M)(1-q) \arrow[ur, dash] &
\end{tikzcd}$$

Proof of II$'$ (comparing projections)

Let p and q be projections in a von Neumann algebra M.

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We say that $p \sim q$ if there is a partial isometry $v \in M$ such that $v^*v = p$ and $vv^* = q$.

The Comparison Theorem

There are projections $x, y \in Z(M)$ such that $px \prec qx$ and $qy \prec py$. In particular, in a factor, either $p \prec q$ or $q \prec p$.

Proof of II$'$ (claim)

We know that M is a I_k factor, so $M \cong M_k(\mathbb{C})$. Let e_{ij} be the matrix units of M (corresponding to the matrix with a 1 in the ij entry and 0's elsewhere).

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Claim

$e_{11}(1-r) \sim \Phi(e_{11})(1-q)$, i.e. there exists a partial isometry $w \in L$ such that $w^*w = \Phi(e_{11})(1-q)$ and $ww^* = e_{11}(1-r)$.

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If not, there exists a projection $z \in Z(L)$ such that $e_{11}(1-r)z \prec \Phi(e_{11})(1-q)z$ by the Comparison Theorem.

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Using the same argument we'll see next, it follows that $e_{ii}(1-r)z \prec \Phi(e_{ii})(1-q)z$ for all $1 \leq i \leq k$, which implies that $(1-r)z \prec (1-q)z$.

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Using the same argument we'll see next, it follows that $e_{ii}(1-r)z \prec \Phi(e_{ii})(1-q)z$ for all $1 \leq i \leq k$, which implies that $(1-r)z \prec (1-q)z$.

However,
$r \sim q \Rightarrow 1-r \sim 1-q \Rightarrow (1-r)z \sim (1-q)z \ \forall z \in Z(L)$ projection. $\Rightarrow\!\Leftarrow$

Proof of II′ (diagram)

We also know that $e_{11} \sim e_{ii}$ for all $1 \leq i \leq k$, as

$$e_{11} = e_{1i}e_{i1} = (e_{i1})^* e_{i1}.$$

Proof of II′ (diagram)

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$$e_{11} = e_{1i}e_{i1} = (e_{i1})^* e_{i1}.$$

So, we get the diagram

$$\begin{tikzcd}
e_{11} \arrow[d, bend left, "e_{i1}"] \\
e_{ii} \arrow[u, bend left, "e_{1i}"]
\end{tikzcd}$$

Proof of II′ (diagram)

$$\begin{tikzcd} e_{11}(1-r) \arrow[d, bend left, "e_{i1}(1-r)"] \\ e_{ii}(1-r) \arrow[u, bend left, "e_{1i}(1-r)"] \end{tikzcd}$$

Proof of II′ (diagram)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
e_{11}(1-r) \arrow[rr, bend left, "w^*"] \arrow[d, bend left, "e_{i1}(1-r)"] & & \Phi(e_{11})(1-q) \arrow[ll, bend left, "w"] \\
e_{ii}(1-r) \arrow[u, bend left, "e_{1i}(1-r)"] & &
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$$\begin{tikzcd}[column sep=huge, row sep=huge]
e_{11}(1-r) \arrow[r, bend left, "w^*"] \arrow[d, bend left, "e_{i1}(1-r)"] & \Phi(e_{11})(1-q) \arrow[l, bend left, "w"] \arrow[d, bend left, "\Phi(e_{i1})(1-q)"] \\
e_{ii}(1-r) \arrow[u, bend left, "e_{1i}(1-r)"] \arrow[r, bend left, "x_i^*"] & \Phi(e_{ii})(1-q) \arrow[l, bend left, "x_i"] \arrow[u, bend left, "\Phi(e_{1i})(1-q)"]
\end{tikzcd}$$

Proof of II$'$ (obtaining v_1)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
e_{11}(1-r) \arrow[r, bend left, "w^*"] \arrow[d, bend left, "e_{i1}(1-r)"] & \Phi(e_{11})(1-q) \arrow[l, bend left, "w"] \arrow[d, bend left, "\Phi(e_{i1})(1-q)"] \\
e_{ii}(1-r) \arrow[u, bend left, "e_{1i}(1-r)"] \arrow[r, bend left, "x_i^*"] & \Phi(e_{ii})(1-q) \arrow[u, bend left, "\Phi(e_{1i})(1-q)"] \arrow[l, bend left, "x_i"]
\end{tikzcd}
$$

Proof of II′ (obtaining v_1)

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e_{11}(1-r) \arrow[r, bend left, "w^*"] \arrow[d, bend left, "e_{i1}(1-r)"] & \Phi(e_{11})(1-q) \arrow[l, bend left, "w"] \arrow[d, bend left, "\Phi(e_{i1})(1-q)"] \\
e_{ii}(1-r) \arrow[u, bend left, "e_{1i}(1-r)"] \arrow[r, bend left, "x_i^*"] & \Phi(e_{ii})(1-q) \arrow[l, bend left, "x_i"] \arrow[u, bend left, "\Phi(e_{1i})(1-q)"]
\end{tikzcd}$$

Let $x_i = e_{i1}(1-r)w\Phi(e_{1i})(1-q)$.

Proof of II′ (obtaining v_1)

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
e_{11}(1-r) \arrow[r, bend left, "w^*"] \arrow[d, bend left, "e_{i1}(1-r)"] & \Phi(e_{11})(1-q) \arrow[l, bend left, "w"] \arrow[d, bend left, "\Phi(e_{i1})(1-q)"] \\
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Let $x_i = e_{i1}(1-r)w\Phi(e_{1i})(1-q)$. Let $v_1 = \sum_{i=1}^{k} x_i$.

Proof of II′ (obtaining v_1)

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e_{11}(1-r) \arrow[rr, bend left, "w^*"] \arrow[d, bend left, "e_{i1}(1-r)"] & & \Phi(e_{11})(1-q) \arrow[ll, bend left, "w"] \arrow[d, bend left, "\Phi(e_{i1})(1-q)"] \\
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\end{tikzcd}
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Let $x_i = e_{i1}(1-r)w\Phi(e_{1i})(1-q)$. Let $v_1 = \sum_{i=1}^{k} x_i$.
v_1 is a partial isometry with $v_1^* v_1 = 1-q$ and $v_1 v_1^* = 1-r$.

Proof of II′ ($v_1^*v_1 = 1 - q$)

$$
\begin{aligned}
v_1^*v_1 &= \sum_{j=1}^{k} x_j^* \sum_{i=1}^{k} x_i \\
&= \sum_{j=1}^{k} \Phi(e_{j1})(1-q)w^*e_{1j}(1-r) \sum_{i=1}^{k} e_{i1}(1-r)w\Phi(e_{1i})(1-q) \\
&= \sum_{i,j=1}^{k} \Phi(e_{j1})(1-q)w^*e_{1j}(1-r)e_{i1}(1-r)w\Phi(e_{1i})(1-q) \\
&= \sum_{i,j=1}^{k} \Phi(e_{j1})(1-q)w^*(\delta_{ji}e_{11}(1-r))w\Phi(e_{1i})(1-q) \\
&= \sum_{i,j=1}^{k} \Phi(e_{j1})(1-q)w^*(\delta_{ji}ww^*)w\Phi(e_{1i})(1-q)
\end{aligned}
$$

Proof of II′ ($v_1^* v_1 = 1 - q$)

$$
\begin{aligned}
&= \sum_{i=1}^{k} \Phi(e_{i1})(1-q)w^* w \Phi(e_{1i})(1-q) \\
&= \sum_{i=1}^{k} \Phi(e_{i1})(1-q)\Phi(e_{11})(1-q)\Phi(e_{1i})(1-q) \\
&= \sum_{i=1}^{k} \Phi(e_{ii})(1-q) \\
&= (1-q)\Phi\left(\sum_{i=1}^{k} e_{ii}\right) \\
&= (1-q)\Phi(1) \\
&= 1-q.
\end{aligned}
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&= (1-q)\Phi\left(\sum_{i=1}^{k} e_{ii}\right) \\
&= (1-q)\Phi(1) \\
&= 1-q.
\end{aligned}
$$

$v_1 v_1^* = 1 - r$ is similar.

Proof of II$'$ (obtaining u)

Let $u = v + v_1$. Then u is a unitary. Indeed:

$$\begin{aligned} uu^* &= (v + v_1)(v + v_1)^* \\ &= vv^* + v_1 v_1^* + vv_1^* + v_1 v^* \\ &= r + 1 - r + 0 + 0 \\ &= 1. \end{aligned}$$

($vv_1^* = 0$, as $\overline{\operatorname{Im} v_1^*} = \operatorname{Im}(1 - q)$ and $v = 0$ here)

$u^* u$ is similar.

Proof of II′ ($\Phi(x) = u^*xu$)

Finally,

$$\begin{aligned} u^* e_{nm} u &= v^* e_{nm} v + v_1^* e_{nm} v_1 \\ &= \Phi(e_{nm}) q + \Phi(e_{nm})(1-q) \\ &= \Phi(e_{nm}). \end{aligned}$$

(the calculation for $v_1^* e_{nm} v_1$ is similar to $v_1^* v_1$)
It follows that $\Phi(x) = u^*xu$ for all $x \in M$.

Proof of II′ ($\Phi(x) = u^*xu$)

Finally,

$$\begin{aligned} u^*e_{nm}u &= v^*e_{nm}v + v_1^*e_{nm}v_1 \\ &= \Phi(e_{nm})q + \Phi(e_{nm})(1-q) \\ &= \Phi(e_{nm}). \end{aligned}$$

(the calculation for $v_1^*e_{nm}v_1$ is similar to $v_1^*v_1$)
It follows that $\Phi(x) = u^*xu$ for all $x \in M$.
Also,

$$||1-u||_2 \leq ||1-v||_2 + ||v_1||_2 \leq 2t + ||1-q||_2 \leq 3t.$$

QED.

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The Basic Construction

The following idea of Christensen's will later become known as Jones' basic construction, as it was used ingeniously in Vaughan Jones' papers that led to a fields medal.

Setting

L is a type II_1 von Neumann algebra and N is a von Neumann subalgebra of L. We have $N \subset L \subset \mathcal{B}(\mathcal{H})$ with $\mathcal{H} = L^2(L, \tau)$.

Since τ is faithful, we also have the embedding $L \subset L^2(L, \tau)$.

Note that L acts on $\mathcal{H}$ via left multiplication, as this is the GNS construction. If we denote by ξ or $\hat{1}$ the vector in $\mathcal{H}$ corresponding to 1 in L, then $x \cdot \hat{1} = \hat{x}$ for all $x \in L$ (where $\hat{x}$ denotes the vector corresponding to x via the embedding $L \subset L^2(L, \tau)$).

e and E

We will denote by e the projection from $L^2(L,\tau)$ onto $L^2(N,\tau)$ and by $E : L \to N$ the conditional expectation from L to N, which is just the restriction of e to L. That is to say:

$$e(\hat{x}) = \widehat{E(x)}.$$

Properties of e and E

- E is well-defined (i.e. it takes values in N) and bimodular: $E(axb) = aE(x)b$ for all $a, b \in N$ and $x \in L$.

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- $x - E(x)$ is, by definition, perpendicular to N for all $x \in L$, i.e. $\tau((x - E(x))^*y) = 0$ for all $x \in L$ and $y \in N$.

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- E is well-defined (i.e. it takes values in N) and bimodular: $E(axb) = aE(x)b$ for all $a, b \in N$ and $x \in L$.
- $x - E(x)$ is, by definition, perpendicular to N for all $x \in L$, i.e. $\tau((x - E(x))^* y) = 0$ for all $x \in L$ and $y \in N$.
- e commutes with N: $ey = ye$ for all $y \in N$. Indeed, for $x \in L$

$$ey\hat{x} = e\widehat{yx} = \widehat{E(yx)} = \widehat{yE(x)} = y\widehat{E(x)} = ye\hat{x}.$$

Properties of e and E (continued)

- For all $x \in L$, $exe = E(x)e$. Indeed, for $z \in L$

$$
\begin{aligned}
exe\hat{z} &= ex\widehat{E(z)} = \widehat{exE(z)} = \widehat{E(xE(z))} \\
&= \widehat{E(x)E(z)} = E(x)\widehat{E(z)} \\
&= E(x)e\hat{z}.
\end{aligned}
$$

Similarly, $eLe = Ne$.

The Basic Construction

We define the basic construction to be

$$L_1 = \langle L, e \rangle = (L \cup \{e\})'' \subset \mathcal{B}(\mathcal{H}),$$

the smallest von Neumann algebra of $\mathcal{B}(\mathcal{H})$ (for $\mathcal{H} = L^2(L, \tau)$) containing L and e.

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- $eL_1e = Ne$.
- If N is a factor, then so is L_1.
- The trace τ of L gives a (faithful, normal, semifinite) trace φ on L_1 with $\varphi(e) = 1$, related to τ by $\varphi(xe) = \tau(x)$ for all $x \in L$.

To be more precise

- The $*$-algebra generated by L, e turns out to be $L + \operatorname{span} LeL$ (using $eLe = Ne$).

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To be more precise

- The $*$-algebra generated by L, e turns out to be $L + \operatorname{span} LeL$ (using $eLe = Ne$).
- We close in the s.o. topology of $\mathcal{B}(\mathcal{H})$: $\overline{L + \operatorname{span} LeL}^{\text{s.o.}} = L_1$.
- The trace is defined by $\varphi(xey) = \tau(xy)$ for all $x, y \in L$ then extended to the closure $\overline{L + \operatorname{span} LeL}^{\text{s.o.}}$ by s.o. continuity.

A note on φ

Note that φ may not take just finite values; $\varphi(1) = \infty$ is possible. That is to say, L_1 is in general a semifinite algebra, so it may not be finite.

If N is a factor, then L_1 is also a factor, and thus it has a unique trace (up to rescaling). So, φ is <u>the</u> trace of L_1.

Example

We've worked with abstract factors so far, so it may be easier to visualize what happens with a concrete (finite dimensional) example.

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Let $L = M_2(\mathbb{C})$ and $D = D_2 \subset M_2(\mathbb{C})$ be the diagonal matrices.

The conditional expectation E is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix},$$

and it corresponds to a projection $e \in \mathcal{B}(L) = \mathcal{B}(M_2(\mathbb{C}))$.

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The conditional expectation E is given by

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and it corresponds to a projection $e \in \mathcal{B}(L) = \mathcal{B}(M_2(\mathbb{C}))$.

We can identify $\mathcal{B}(M_2(\mathbb{C}))$ with $M_4(\mathbb{C})$, with $L = M_2(\mathbb{C})$ embedded in it via left multiplication.

Example (continued)

It can help to think of $M_4(\mathbb{C})$ as $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ and $L = M_2(\mathbb{C}) \otimes I \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$.

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It can help to think of $M_4(\mathbb{C})$ as $M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ and $L = M_2(\mathbb{C}) \otimes I \subset M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$.

In this case, $e \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C})$ is $e_{11} \otimes e_{11} + e_{22} \otimes e_{22} =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and $\langle L, e \rangle = M_2(\mathbb{C}) \otimes D_2$.

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and $\langle L, e \rangle = M_2(\mathbb{C}) \otimes D_2$.

Lastly, $\varphi = \frac{1}{2} \mathrm{Tr}_{M_4(\mathbb{C})}$ (so $\varphi(e) = 1$).

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Theorem I

Theorem (I)

If $M, N \subset L$ with M finite dimensional, N a II_1 factor, and $M \overset{\delta}{\subset} N$ for some $\delta < 1/\sqrt{2}$, then there exist $\Phi : M \to N$, Φ an isomorphism from M to a von Neumann subalgebra $\Phi(M)$ of N, and a fixed constant c such that for all $x \in M_1$

$$||\Phi(x) - x||_2 \leq c\delta^{1/2}$$

($c = 1050$ should work).

Proof of I (setup)

We have $M, N \subset L$, M finite dimensional, N II_1 factor, $M \overset{\delta}{\subset} N$ with $\delta > 0$ small.

We want to construct $\Phi : M \to N$ morphism (isomorphism onto $\Phi(M)$) with $||\Phi(x) - x||_2 \leq c\delta^{1/2}$ for all $x \in M_1$, for some constant c (specifically $c = 1050$).

Proof of I (idea)

- Construct $L_1 = \langle L, e \rangle$ with e projection from $L^2(L, \tau)$ to $L^2(N, \tau)$. Since N is a factor, L_1 is also. We know e commutes with N, and since $M \overset{\delta}{\subset} N$, we can show that e "almost" commutes with M. In other words, $||e - u^*eu||_{2,\varphi}$ is small for $u \in \mathcal{U}(M)$.

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- We can find k of minimal norm $|| \ ||_{2,\varphi}$ in $\overline{\mathrm{co}}_M^{|| \ ||_{2,\varphi}}(e)$, with $||e-k||_{2,\varphi}$ small and $k \in M'$.

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- We can find k of minimal norm $|| \; ||_{2,\varphi}$ in $\overline{\mathrm{co}}_M^{|| \; ||_{2,\varphi}}(e)$, with $||e-k||_{2,\varphi}$ small and $k \in M'$.
- From k, we construct one of its spectral projections q, with $||e-q||_{2,\varphi}$ small and $q \in M'$.

Proof of I (idea)

- Construct $L_1 = \langle L, e\rangle$ with e projection from $L^2(L,\tau)$ to $L^2(N,\tau)$. Since N is a factor, L_1 is also. We know e commutes with N, and since $M \overset{\delta}{\subset} N$, we can show that e "almost" commutes with M. In other words, $||e - u^*eu||_{2,\varphi}$ is small for $u \in \mathcal{U}(M)$.
- We can find k of minimal norm $||\ ||_{2,\varphi}$ in $\overline{\mathrm{co}}_M^{||\ ||_{2,\varphi}}(e)$, with $||e-k||_{2,\varphi}$ small and $k \in M'$.
- From k, we construct one of its spectral projections q, with $||e-q||_{2,\varphi}$ small and $q \in M'$.
- Since L_1 is a factor with $e, q \in L$, the Comparison Theorem tells us that $q \prec e$ or $q \succ e$.
  - If $q \overset{v}{\prec} e$, we construct $\Phi(x)$ from v^*xv.
  - If $q \succ e$, then $q \geq r \sim e$, and we use r as above ($r \sim e$, so $r \prec e$).

Proof of I (the basic construction)

Consider the basic construction $L_1 = \langle L, e \rangle$, where e is the projection from $L^2(L, \tau)$ to $L^2(N, \tau)$.

- N is a factor, so L_1 is a factor.
- L_1 comes with a semifinite trace φ satisfying $\varphi(e) = 1$.

Proof of I (e almost commutes with M)

We know that e commutes with N, so $ue = eu$ for all $u \in \mathcal{U}(N)$. Hence $u^*eu = e$ for all $u \in \mathcal{U}(N)$.

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Since $M \overset{\delta}{\subset} N$, we expect that for $u \in \mathcal{U}(M)$, u^*eu is “close” to e. Indeed:

$$\begin{aligned}
||e - u^*eu||_{2,\varphi}^2 &= \varphi((e - u^*eu)^*(e - u^*eu)) \\
&= \varphi(e - eu^*eu - u^*eue + u^*eu) \\
&= 2\varphi(e - eu^*eue) \ (\varphi \text{ trace}) \\
&= 2\varphi(e - E(u^*)E(u)e) \ (exe = E(x)e) \\
&= 2\varphi((1 - E(u^*)E(u))e) \\
&= 2\tau(1 - E(u^*)E(u)) \ (\varphi(xe) = \tau(x)) \\
&\leq 2\delta^2.
\end{aligned}$$

Proof of I (e almost commutes with M)

Thus, $||e - u^*eu||_{2,\varphi} \leq \sqrt{2}\delta$. For the last step,

$$2\tau(1 - E(u^*)E(u)) \leq 2\delta^2,$$

we used

- $\tau(1 - E(u^*)E(u))^{1/2} = ||E(u) - u||_{2,\tau}$
- $||E(u) - u||_{2,\tau} \leq \delta$, which follows from $M \overset{\delta}{\subset} N$ and the fact that $E(u)$ minimizes the $||\ ||_{2,\tau}$ distance between u and N

Proof of I (closed convex hull)

Consider now the closed convex hull $\overline{\mathrm{co}}_M^{\|\ \|_{2,\varphi}}(e) =$

$$\overline{\left\{ \sum_{i=1}^{n} c_i u_i^* e u_i : n \geq 1, u_i \in \mathcal{U}(M), c_i > 0, \sum_{i=1}^{n} c_i = 1 \right\}}^{\|\ \|_{2,\varphi}},$$

and let k be the unique element of minimal norm $\|\ \|_{2,\varphi}$ in it.

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and let k be the unique element of minimal norm $\|\ \|_{2,\varphi}$ in it.

Since $\|e - u^* e u\|_2 \leq \sqrt{2}\delta$ for all $u \in \mathcal{U}(M)$, it follows that $\|e - k\|_{2,\varphi} \leq \sqrt{2}\delta$.

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Consider now the closed convex hull $\overline{\mathrm{co}}_M^{|| \, ||_{2,\varphi}}(e) =$

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and let k be the unique element of minimal norm $|| \, ||_{2,\varphi}$ in it.

Since $||e - u^* e u||_2 \leq \sqrt{2}\delta$ for all $u \in \mathcal{U}(M)$, it follows that $||e - k||_{2,\varphi} \leq \sqrt{2}\delta$.

Since $||u^* k u||_{2,\varphi} = ||k||_{2,\varphi}$ and k is the unique element of minimal norm, it follows that $u^* k u = k$. Thus, $k \in M'$.

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Since $||u^* k u||_{2,\varphi} = ||k||_{2,\varphi}$ and k is the unique element of minimal norm, it follows that $u^* k u = k$. Thus, $k \in M'$.

Also, $k \in \langle M, e \rangle$ (with $\langle M, e \rangle = (M \cup \{e\})''$).

Proof of I (spectral projection)

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$$\gamma = 2^{1/4}\delta^{1/2}(1 - 2^{1/4}\delta^{1/2})^{-1}.$$

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$$\gamma = 2^{1/4}\delta^{1/2}(1 - 2^{1/4}\delta^{1/2})^{-1}.$$

Since $\varphi(e) = 1$, we can show that $|1 - \varphi(q)| \leq \gamma^2$.

Proof of I (The Comparison Theorem)

Since $k \in M' \cap \langle M, e \rangle$, so is q:

$$q \in M' \cap \langle M, e \rangle \subset M' \cap L_1.$$

Since L_1 is a factor and q, e are projections in L_1, the Comparison Theorem tells us that $q \prec e$ or $e \prec q$.

Proof of I (case 1: $q \prec e$)

If $q \prec e$, then (by definition) $q \sim q' \leq e$ for some projection q'.

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If $q \prec e$, then (by definition) $q \sim q' \leq e$ for some projection q'. We have

$$\begin{aligned} ||q - q'||_{2,\varphi} &\leq ||q - e||_{2,\varphi} + ||e - q'||_{2,\varphi} \\ &\leq \gamma + \varphi(e - q')^{1/2} \\ &= \gamma + (1 - \varphi(q))^{1/2} \\ &\leq 2\gamma. \end{aligned}$$

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Since $q \sim q'$, there is a partial isometry $v \in L_1$ such that $v^*v = q'$ and $vv^* = q$. Using one of Christensen's lemmas, since $||q - q'||_{2,\varphi} \leq 2\gamma$, we can choose v with $||v - q'||_{2,\varphi} \leq 12\gamma$.

Proof of I (defining Φ)

Define $\Phi : M \to N$ by $\Phi(x) = v^* x v$ for all $x \in M$. This can be verified to be a homomorphism.

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Really, this takes values in Ne, rather than N:

$$v = ve \Rightarrow v^*Mv = ev^*Mve \subset eL_1e = Ne.$$

Thankfully, Ne is isomorphic to N via $n \overset{\alpha}{\mapsto} ne$ (since e commutes with N). So, $\Phi = \alpha \circ \operatorname{Ad} v^*$

$$M \overset{\operatorname{Ad} v^*}{\to} Ne \overset{\alpha}{\to} N$$

Proof of I (inequality)

Using the inequalities that we have so far, we can show that $||\Phi(x) - x||_2 \leq \beta\gamma$ for all $x \in M_1$, for some fixed β ($\beta = 105$ should work) (for δ sufficiently small, $\gamma < 10\delta^{1/2}$).

Proof of I (case 2: $e \prec q$)

If $e \prec q$, then $e \sim r \leq q$ with $r \in M' \cap L_1$ and $\varphi(r) = \varphi(e) = 1$. As above,

$$\begin{aligned} ||e - r||_{2,\varphi} &\leq ||e - q||_{2,\varphi} + ||q - r||_{2,\varphi} \\ &\leq \gamma + (1 - \varphi(q))^{1/2} \\ &\leq 2\gamma. \end{aligned}$$

So, we can use r as in the previous case, instead of q, to construct $\Phi : M \to N$ ($r \sim e$, so in particular $r \prec e$). QED.

Table of Contents

The Jones Tower

In 1983, Jones introduced the basic construction, along with a notion of index $[M : N]$ (for $N \subset M$).

Under certain conditions, the basic construction can be repeated to obtain a Jones tower:

$$N \subset M \overset{e_0}{\subset} M_1 \overset{e_1}{\subset} M_2 \overset{e_2}{\subset} \cdots$$

The Standard Invariant

By intersecting the Jones tower with commutants, we obtain the Standard Invariant:

$$\begin{array}{ccccccccc} \mathbb{C} & = & N' \cap N & \subset & N' \cap M & \subset & N' \cap M_1 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \cup \\ & & \mathbb{C} & = & M' \cap M & \subset & M' \cap M_1 & \subset & \cdots \end{array}$$

Commuting Squares

A square of inclusions

$$\begin{array}{ccc} A & \subset & B \\ \cup & & \cup \\ C & \subset & D \end{array}$$

such that $A \ominus C \perp D \ominus C$ is called a commuting square (equivalently, $E_A E_D = E_D E_A = E_C$).

Thank you!

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